Using the mediation operator to express the new points in the scaffolding in terms of the original control points produces, for the quadratic case

$$\mathbf{P}_{01} = \alpha\mathbf{P}_0+(1-\alpha)\mathbf{P}_1, \mathbf{P}_{12} = \alpha\mathbf{P}_1+(1-\alpha)\mathbf{P}_2, \mathbf{P}_{012} = \alpha^2\mathbf{P}_0+2\alpha(1-\alpha)\mathbf{P}_1+(1-\alpha)^2\mathbf{P}_2,$$

where α is any value in the range $[0,1]$. We can therefore write

$$\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_{01}\\ \mathbf{P}_{012}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ \alpha & 1-\alpha & 0\\ \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2\end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix},$$
$$\begin{pmatrix}\mathbf{P}_{012}\\ \mathbf{P}_{12}\\ \mathbf{P}_2\end{pmatrix} = \begin{pmatrix}\alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2\\ 0 & \alpha & 1-\alpha\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix},$$

for the left and right segments, respectively.

▸ **Exercise 01:** Use the mediation operator to calculate the scaffolding for the cubic case (four control points). Use $\alpha = 1/2$ and write the results in terms of matrices, as above.

Answer Figure 01 shows the new points. For an arbitrary α their values are

$$\mathbf{P}_{01} = \alpha\mathbf{P}_0 + (1-\alpha)\mathbf{P}_1, \quad \mathbf{P}_{12} = \alpha\mathbf{P}_1 + (1-\alpha)\mathbf{P}_2, \quad \mathbf{P}_{23} = \alpha\mathbf{P}_2 + (1-\alpha)\mathbf{P}_3,$$
$$\mathbf{P}_{012} = \alpha^2\mathbf{P}_0 + 2\alpha(1-\alpha)\mathbf{P}_1 + (1-\alpha^2)\mathbf{P}_2, \quad \mathbf{P}_{123} = \alpha^2\mathbf{P}_1 + 2\alpha(1-\alpha)\mathbf{P}_2 + (1-\alpha)^2\mathbf{P}_3,$$
$$\mathbf{P}_{0123} = \alpha^3\mathbf{P}_0 + 3\alpha^2(1-\alpha)\mathbf{P}_1 + 3\alpha(1-\alpha)^2\mathbf{P}_2 + (1-\alpha)^3\mathbf{P}_3.$$

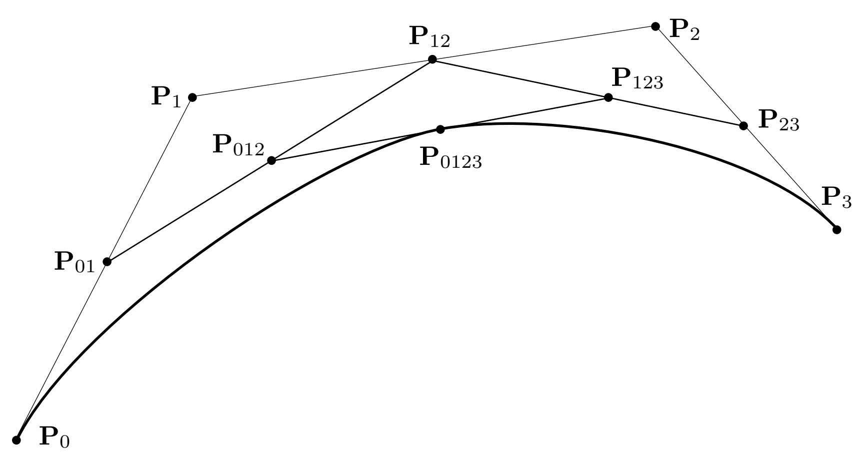

Figure 01: Scaffolding for $k = 3$.

Using matrix notation, this can be expressed as,

$$\begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_{01} \\ \mathbf{P}_{012} \\ \mathbf{P}_{0123} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1-\alpha & 0 & 0 \\ \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 & 0 \\ \alpha^3 & 3\alpha^2(1-\alpha) & 3\alpha(1-\alpha)^2 & (1-\alpha)^3 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = \mathbf{M}_L(\alpha)\mathbf{G},$$

$$\begin{pmatrix} \mathbf{P}_{0123} \\ \mathbf{P}_{123} \\ \mathbf{P}_{23} \\ \mathbf{P}_3 \end{pmatrix} = \begin{pmatrix} \alpha^3 & 3\alpha^2(1-\alpha) & 3\alpha(1-\alpha)^2 & (1-\alpha)^3 \\ 0 & \alpha^2 & 2\alpha(1-\alpha) & (1-\alpha)^2 \\ 0 & 0 & \alpha & 1-\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = \mathbf{M}_R(\alpha)\mathbf{G},$$

where $\mathbf{G}$ is the column consisting of the four original control points of the segment. Notice that the elements of each row of matrices $\mathbf{M}_L(\alpha)$ and $\mathbf{M}_R(\alpha)$ are barycentric. For the special case $\alpha = 0.5$, these expressions reduce to

$$\begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_{01} \\ \mathbf{P}_{012} \\ \mathbf{P}_{0123} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 8 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{P}_{0123} \\ \mathbf{P}_{123} \\ \mathbf{P}_{23} \\ \mathbf{P}_3 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix}$$

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