

1 Perspective in Curved Objects

This document complements the material in section 3.3 of *Transformations and Projections in Computer Graphics* (Springer Verlag, 2006, ISBN 1-84628-392-2). It discusses techniques for drawing curved objects in perspective. There are many books, mostly for artists, draftsmen, and architects, that discuss perspective and describe methods for drawing objects in perspective. Unfortunately, these books generally employ cubes, or cubical objects, as examples, and therefore create the wrong impression that only such objects can be drawn in perspective. Experienced artists, illustrators, and other people (such as engineers and architects) who draw both curved and cubical objects, know that even curved objects, even objects that lack any straight lines and flat faces, can be drawn in perspective. This document explains simple techniques and approaches to the perspective drawing of arbitrary objects. For general references on this topic and more examples, see [Hulsey 08] and [Robertson 08].

We start with a general 1-point perspective grid. Figure 1a shows a rectangle bounded by two horizontal lines and the vertical lines a and b . We want to construct an adjacent rectangle of the same width. Two diagonals are drawn (shown in dashed) to locate the center of the rectangle, and a short horizontal line c is drawn to locate point B , the center of line b . Finally, a line is drawn from point A through point B , to determine point C , which becomes the bottom-right corner of the new rectangle. This simple technique is now applied to construct a perspective grid.

Figure 1b shows two boundary lines that converge to a vanishing point VP . Two vertical lines a and b are selected arbitrarily (it is useful to assume that the distance between them is one unit) to create a trapezoid. We draw the two diagonals (in dashed) and construct the line from the center of the trapezoid to the vanishing point.

In part (c) of the figure we construct a line from point A through point B (the center of line b) to obtain point C , which determines the position of the next vertical, c , and thus the next trapezoid. Notice that the distance between b and c is shorter than the distance between a and b , because line c is farther away from the observer. This is why the distances between the consecutive verticals in part (d) of the figure are diminishing, an effect called foreshortening. Part (e) of the figure shows two 1-point perspective grids with different orientations relative to a horizon line.

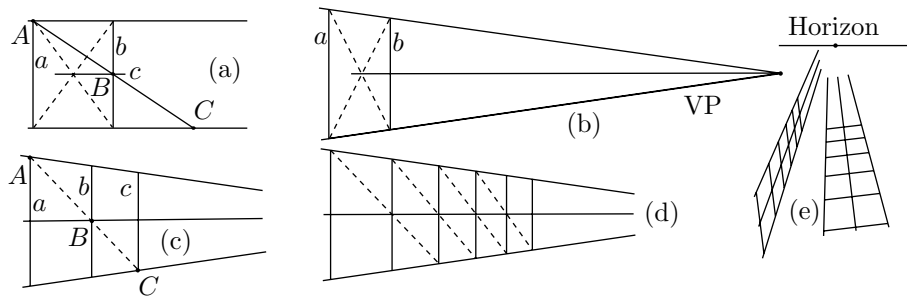


Figure 1: A 1-Point Perspective Grid.

A 2-point perspective grid is constructed in a similar process, as illustrated by Figure 2. It is clear that each square (or rectangle) in the original coordinate system is projected to a quadrilateral.

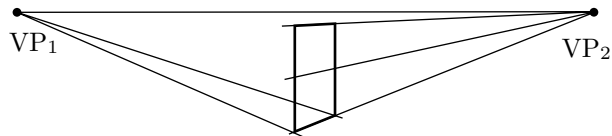


Figure 2: A Quadrilateral in a 2-Point Perspective Grid.

We therefore conclude that each rectangle (or square) in the original coordinate system is mapped to a trapezoid in a 1-point perspective grid and to a quadrilateral in a 2-point perspective grid. Thus, given an arbitrary figure, we generate its 1-point (or 2-point) perspective projection by first constructing a bounding rectangle around it and then mapping every point in this rectangle to the trapezoid (or quadrilateral) given by the perspective grid. Figure 3 shows how the mathematical expressions of this mapping are developed.

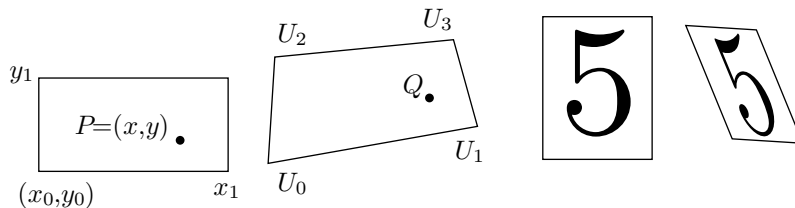


Figure 3: Mapping a Rectangle to a Quadrilateral.

Given a point $P = (x_0, y_0)$ in a rectangle, its projection Q in the quadrilateral defined by the four points $U_i = (u_i, v_i)$ is computed in the following three steps:

$$a = \frac{x - x_0}{x_1 - x_0}, \quad b = \frac{y - y_0}{y_1 - y_0}.$$

$$b_1 = (1 - b)U_0 + bU_2, \quad b_2 = (1 - b)U_1 + bU_3.$$

$$Q = (1 - a)b_1 + ab_2.$$

The quantity a is the relative distance of P from the left edge of the rectangle. It is a number in the interval $[0, 1]$. Similarly, b is the relative distance of P from the bottom of the rectangle. Once b is known, it is used to compute points b_1 and b_2 (these are points, not numbers). Point b_1 is located on the left edge of the quadrilateral, at relative distance b from U_0 and similarly for point b_2 . Finally, point Q is computed on the line connecting b_1 and b_2 at a relative distance a from the left.

Many drawing and illustration programs can perform this mapping automatically, as illustrated by the large digit **5** in the figure.

Symmetric objects. The special case of a symmetric object is important and is considered next. Many common, important objects feature some type of symmetry. Animals, kitchenware (plates, pots, cups, forks), vehicles, tools, and many furniture exhibit at least left-right symmetry, while wheels and other circular objects feature higher symmetries. To draw an object with left-right symmetry, it is possible to draw one-half of the object, select several strategically-placed points on it, and employ simple geometric constructs to transfer those points to the “other side,” where they can be used to draw the other half of the object. Figure 4 illustrates this technique.

In part (a) we start with a simple 1-point perspective grid consisting of two “horizontal” lines that converge to a vanishing point and three verticals. The result is two quadrilaterals. We draw half of a symmetric curve in one quadrilateral and use simple geometric construction to mirror it in the other quadrilateral. Dashed lines a and b are symmetric and line (a) intersects our curve at a point O . We therefore construct line l that converges to the vanishing point and locate point P , the mirror image of O , at the intersection of l and b . Similarly, lines d and e are employed to locate point Q and line m is used to locate point R . More mirror points can be located in this way, until there are enough points to complete the missing half of the curve (Figure 4b). It is obvious that the two halves, which are symmetric, have very different shapes in a perspective drawing.

In Figure 4c, we draw line n to start another 1-point perspective grid and draw one-half of another symmetric curve. In part (d), two diagonals are drawn to determine the center of a quadrilateral, from which point we draw line n to the vanishing point. Another diagonal locates point S , from which we draw vertical p . This completes the matching quadrilateral. In part (e) of the figure we determine two strategic points, using the same techniques as in part (a), and the complete curve is shown in part (f) of the figure.

Figure 5 illustrates this technique in 2-point perspective. Part (a) of the figure shows two groups of “horizontal” lines that converge to two vanishing points. A few “verticals” are also shown. This construction provides the background grid for the drawing. In part (b) we see a vertical curve that immediately makes it clear that this is going to be the drawing of a car. Part (c) shows one-half of the bottom of the car (a horizontal curve) with point A selected. The bottom is a symmetric curve, but it is clear from the figure that while it is easy to draw the half curve that we see (the one closer to us), it is much harder to draw its symmetric counterpart. This is true for both freehand drawing and drawing done with special graphics software (the drawings shown here were done in a very old version of Adobe Illustrator). Thus, our immediate problem is to transfer point A to the “other side” of the drawing (to mirror it).

We start with the simple construction of part (d). A vertical is drawn from point A and line a is drawn to intercept vertical v at point B . Notice that lines a and b are not parallel; they meet at VP_1 . Finally, in part (e) we draw the two diagonals shown in dashed to determine the center of the quadrilateral, draw line c from the center toward VP_1 , and draw a line from the top-right corner of the quadrilateral through point C to intercept line b at point D , which is the mirror image of point A .

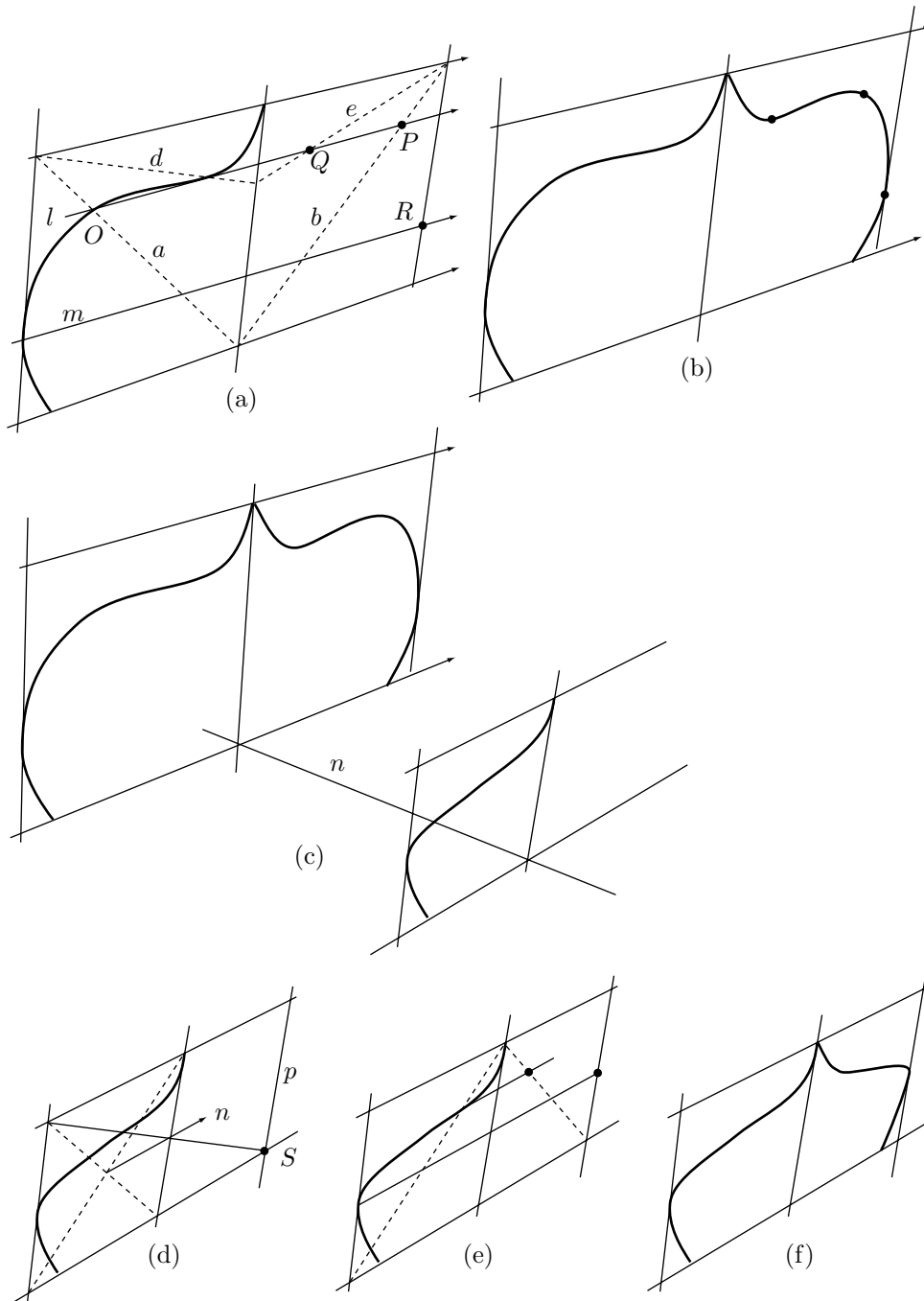


Figure 4: Mirroring Points.

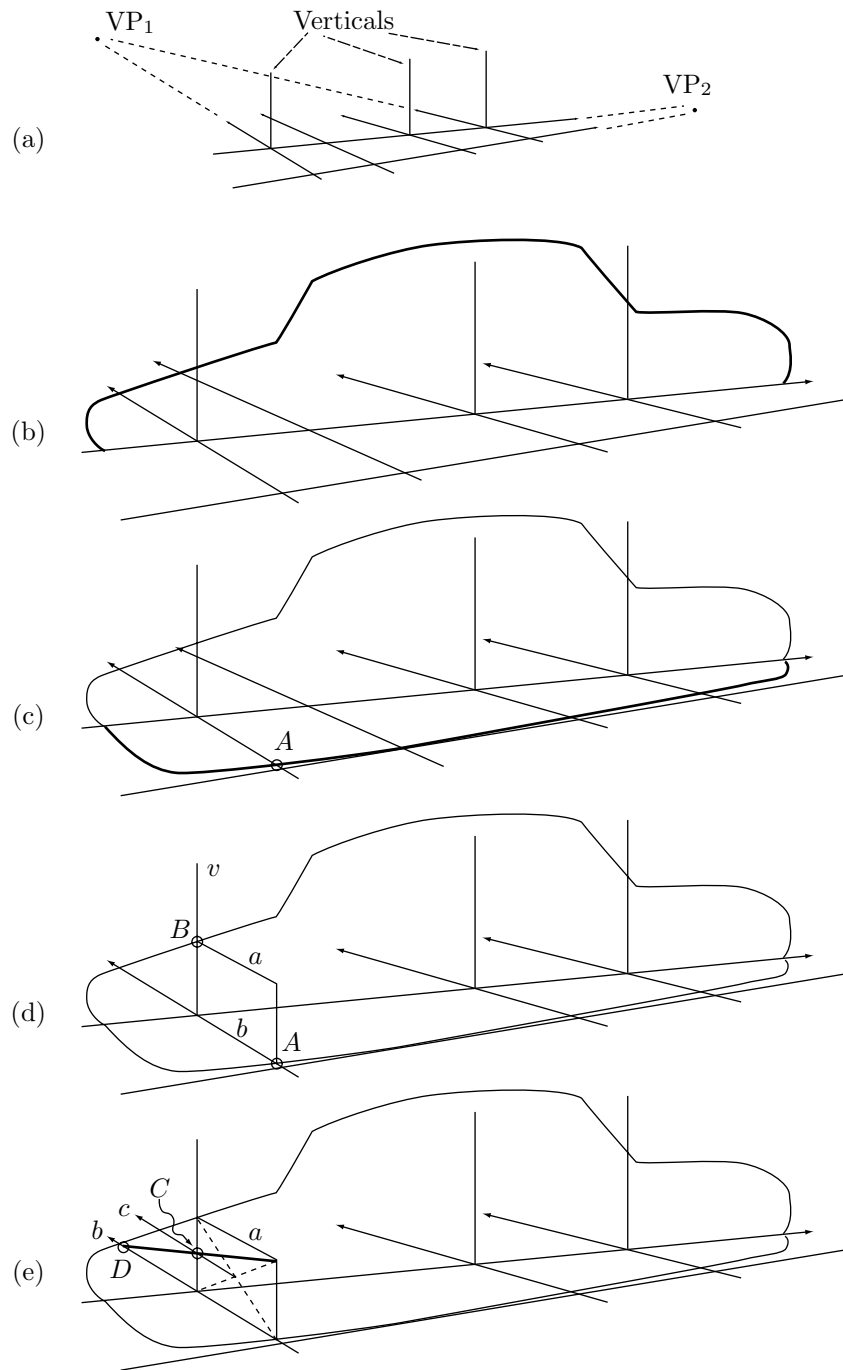


Figure 5: Mirroring a Point.

Note. The height of the vertical drawn from point A in part (d) was chosen such that point B is both on vertical v and on the center profile of the car. It is important to realize that the height of this vertical can be chosen somewhat arbitrarily and that point B does not have to lie on the center profile. In fact, it seems that the best choice is for the resulting quadrilateral to be as close to a rectangle as possible, because this makes it easier to determine an accurate center point in part (e).

Figure 6 applies this technique to point E of the new curve w . This is a “transverse” curve describing half of the lateral profile of the car, and the construction shows how point F , the mirror image of E , is determined.

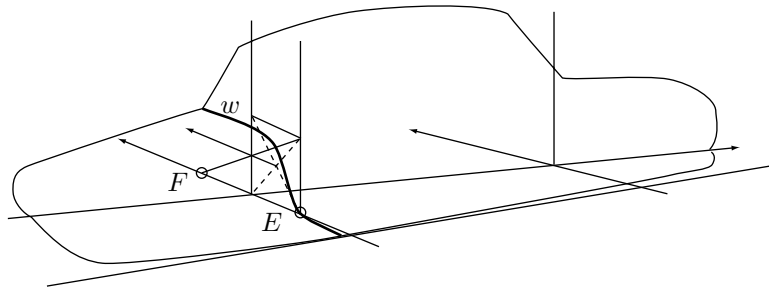


Figure 6: Mirroring a Point.

Circles

Circles are rare in nature (are crop circles natural?), but common in man-made objects (Figure 8). When it comes to drawing objects with circles, the guiding principle is that the perspective projection of a circle is an ellipse (see Figure 8 and [Bartlett 08] for illustrations and [Moore 89] for a proof). We start with a few facts about the ellipse.

The term “ellipse” is derived from the Greek $\epsilon\lambda\lambda\epsilon\iota\psi\iota\varsigma$, meaning absence (see [Heath 81] for an explanation of this). An ellipse is the locus of all the points for which the sum of the distances to two fixed points, called the foci, is constant. An ellipse centered on the origin with foci at points $(-c, 0)$ and $(c, 0)$ is called canonical (Figure 7a). Its implicit representation is $(x/a)^2 + (y/b)^2 = 1$, where $2a$ and $2b$ are the major and minor axes, respectively.

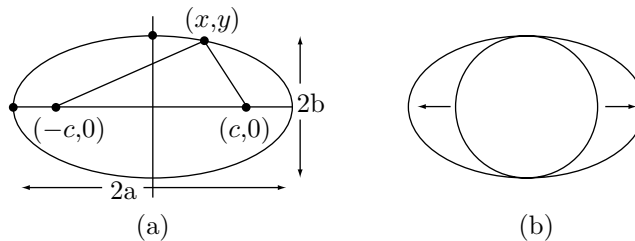


Figure 7: A Canonical Ellipse.



Figure 8: Natural and Artificial Circles.

It's an ellipse.

—Gordon Eklund and Gregory Benford, *If the Stars Are Gods* (1977).

The ellipse can be represented parametrically by means of

$$\mathbf{E}(t) = (a \cos(2\pi t), b \sin(2\pi t)), \quad 0 \leq t \leq 1,$$

or

$$\mathbf{E}(t) = \left(a \left(\frac{1-t}{1+t} \right), b \left(\frac{2\sqrt{t}}{1+t} \right) \right), \quad 0 \leq t \leq \infty.$$

When $a = b$, these expressions reduce to the parametric representations of a circle.

The eccentricity of the ellipse measures how much it deviates from a circle. It is defined as $e = c/a$. For a circle, $e = 0$. When $e = 1$, the ellipse reduces to a line from $(-c, 0)$ to $(c, 0)$. The eccentricity of the Earth's orbit around the Sun is $\approx 1/60$. However, when an ellipse is drawn as the perspective projection of a circle, it is the degree, not the eccentricity, that is used as the measure of deviation. A circle is a 90° ellipse, while a straight line is a 0° ellipse. Thus, the degree of an ellipse E equals $1 - \theta$, where θ is the angle through which the original circle had to be rotated to look like E .

When an area is scaled (expanded or shrunk), the determinant of the scaling matrix equals the scaling factor. This can be used to determine the area πab of the ellipse. The equations of the circle and the ellipse are $x^2 + y^2 = R^2$ and $(x/a)^2 + (y/b)^2 = 1$. Therefore, if point (x, y) is on the circle, it can be transformed to the ellipse by the scaling transformation

$$\begin{pmatrix} a/R & 0 \\ 0 & b/R \end{pmatrix}.$$

The determinant of this matrix equals ab/R^2 , so the area of the ellipse equals the circle area times ab/R^2 or $\pi R^2 \times ab/R^2 = \pi ab$.

A circle no doubt has a certain appealing simplicity at first glance, but one look at an ellipse should have convinced even the most mystical of astronomers that the perfect simplicity of the circle is akin to the vacant smile of complete idiocy. Compared to what an ellipse can tell us, a circle has little to say. Possibly our own search for cosmic simplicities in the physical universe is of this circular kind—a projection of our uncomplicated mentality on an infinitely intricate external world.

— Eric Temple Bell, *Mathematics: Queen and Servant of Science*.

Drawing an ellipse with graphics software is easy. All drawing and illustration programs have a special tool for drawing circles. Often, this tool can also be used to draw ellipses, but in any case an ellipse can be drawn by starting with a circle and scaling it in one dimension (Figure 7b).

Figure 9 illustrates how an ellipse behaves as the 1-point perspective projection of a circle. Part (a) of the figure shows an ellipse with its two main axes and it is clear that the center point (defined as the intersection of the two diagonals of the bounding box) is located at the intersection of the two axes. Also, the two extreme points above and below the center point divide the ellipse into two equal parts. In order to fit this ellipse in a 1-point perspective grid, we reshape its bounding box and change it from a rectangle to a trapezoid. The two dashed diagonals in part (b) of the figure determine the new (perspective) center of the ellipse, and it is obvious that this center is no longer located on the major axis. Also, the two extreme points above and below the center now divide the ellipse into two unequal parts. Notice, however, that the minor axis still divides the ellipse into two identical parts, which is one reason why the minor axis of the ellipse is most important in perspective drawing.

He [Bean] had already read all the major writers and many of the minor ones and knew the important campaigns backward and forward, from both sides.

—Orson Scott Card, *Ender's Shadow*.

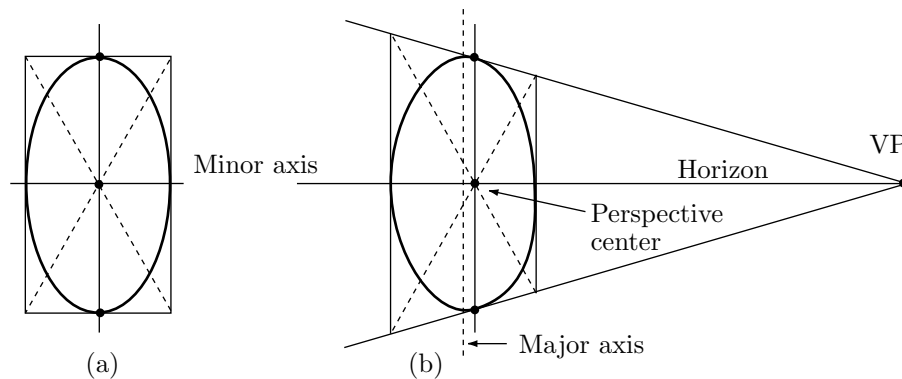


Figure 9: The Ellipse in 1-Point Perspective.

Thus, we conclude that the perspective projection of a given circle is the ellipse

that satisfies the following conditions: (1) Its minor axis points in the direction of a vanishing point and (2) it touches the trapezoid at the centers of its sides and is tangent to each side.

Next, we extend the 1-point perspective grid by constructing more verticals. Vertical a is drawn at an arbitrary location, and then a line is drawn (in dashed) from point A through point B , to intercept the bottom border of the grid at point C , which specifies the next vertical b . Figure 10 shows how this process is repeated four more times, to end up with five trapezoids, which are the projections of five identical squares, each the bounding box of a circle. The figure also shows ellipses drawn in the first and last trapezoids. The principle of drawing these ellipses is to keep their minor axes on the horizon (i.e., facing the vanishing point).

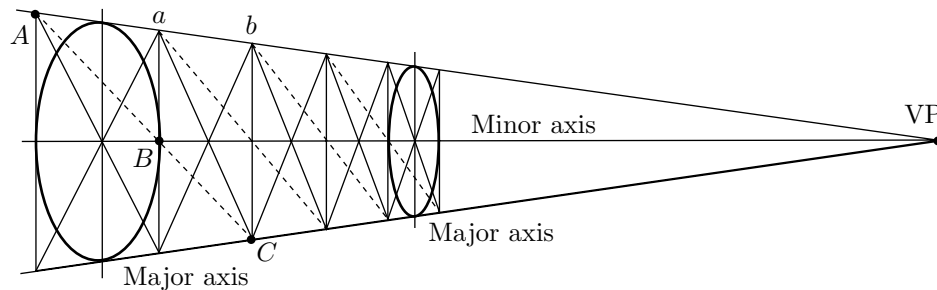


Figure 10: Two Ellipses in 1-Point Perspective.

We are now ready to extend this technique to 2-point perspective. Figure 11a depicts a horizon line with two vanishing points and a 1-point perspective grid that converges to VP_2 . We select the two extreme trapezoids and draw lines from their centers to VP_1 . These lines become the minor axes of the ellipses shown in part (b) of the figure. The major axes are simply the lines perpendicular to the minor axes and are also shown.

The Cutters had major as well as minor subjects for dispute. The chief of these was the question of inheritance.

—Willa Cather, *My Antonia* (1918).

It is also important to discuss how such nonstandard ellipses can be drawn. Notice that the ellipses of Figure 11b touch the centers of (and are tangent to) the four sides of the trapezoid, as indicated by the four small triangles. Just drawing a vertical ellipse and then rotating it, as illustrated by the dashed ellipse of Figure 11c, does not produce the correct result. In Adobe Illustrator (or a similar drawing program), it is better to start with a vertical ellipse and then drag each of its four anchor points to the center of the trapezoid's side nearest to it (indicated by the small triangles in Figure 11b,c) and adjust the tangent at the anchor point to be parallel to that side, as shown in the figures. A different approach to drawing ellipses is discussed in [Bartlett 08].

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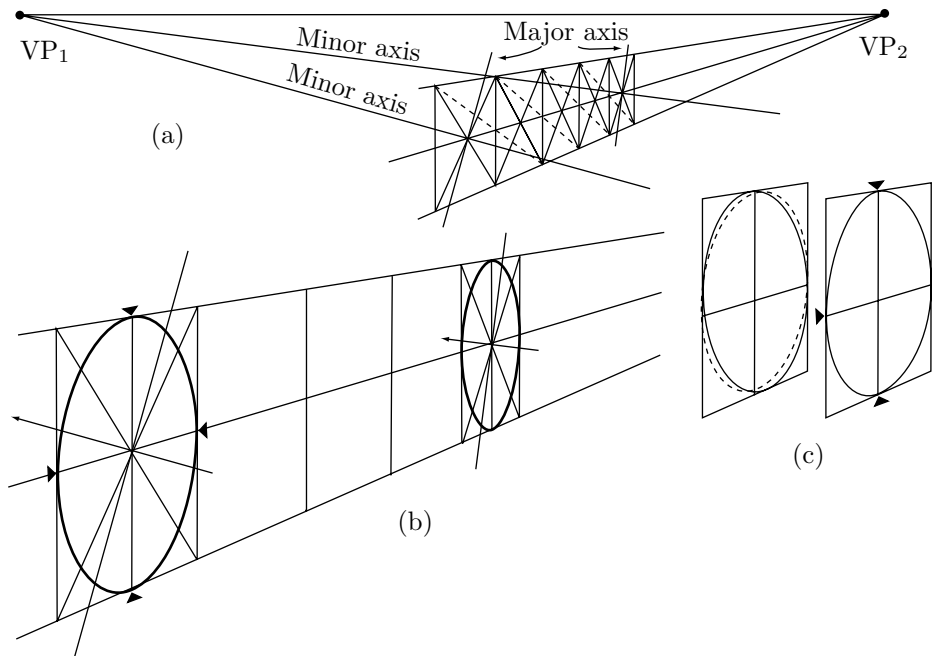


Figure 11: Ellipses in 2-Point Perspective.

References:

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Hulsey, Kevin (2008) is <http://www.khulsey.com/student.html>.

Moore, Charles G. (1989) "To View an Ellipse in Perspective," *The College Mathematics Journal*, **20**(2):134–136, March.

Robertson, Scott (2008) is <http://drawthrough.com/dvds.php>.

Questions, comments, and error corrections should be sent to dsalomon@csun.edu.

Recently someone emailed me asking me to explain perspective. I soon realised that explaining perspective in an email was like getting a marshmallow into a piggy bank but I did feel that I could explain the basics on a web page if I included lots of pictures.

—W. G. Anderson, 2007