Elementary Matrix Representation of Some Commonly Used Geometric Transformations in Computer Graphics and Its Applications

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Abstract

By using an extension of Desargues theorem, and the extended Desarguesian configuration thereof, the analytical definition for the stereohomology geometric transformation in projective geometry is proposed in this work. A series of such commonly used geometric transformations in computer graphics as central projection, parallel projection, centrosymmetry, reflection and translation transformation, which can be included in stereohomology, are thus analytically defined from this point of view. It has been proved that, the transformation matrices of stereohomology are actually equivalent to the existing concept in numerical analysis: elementary matrices. Based on the meaning of elementary matrices in projective geometry thus obtained, a novel approach of 3D reconstructing objects from multiple views is proposed together with some of the concepts, principles and rules for its applications in computer graphics.

Keywords:

Elementary matrix; Homogeneous coordinates; Desargues theorem; Perspective projection; 3D reconstruction; Projective geometry.

1. Introduction

As is well known that, such geometric transformations as translation transformation, perspective projection in computer graphics, can be represented as square transformation matrices only through the homogeneous coordinate representation.

Yet it is surprising to see that presently there is no rigorous definition for the geometric transformations represented by square matrices based on homogeneous coordinates. Ideally, a geometric transformation and its transformation matrix, should be defined and identified reference coordinate system independently, while the conventional representation method and the definitions in computer graphics, at least for geometric transformation matrices, are not in such a way.

Actually, the conventional definitions for the commonly used geometric transformations and their transformation matrices, are not perfect, or even not theoretically rigorous in the following two aspects:

First, by using homogeneous coordinates, we assume that the geometric transformations are defined in projective space not in Euclidean space. As we know, in most of the current computer graphics textbooks, the geometric transformations, for example, central projection, and parallel projection, are actually defined in Euclidean space by using the concepts of measure or distance directly or implicitly which will be inapplicable in projective space.

Second, conventionally, the geometric transformation matrices can also be established for perspective projection, translation transformation and so forth, based on homogeneous coordinates representation in special reference coordinate systems. While mainly due to the limitations of the traditional representations, there is actually no simple and straightforward reference-coordinate-system independent method for transformation matrix establishing.

Since most of the geometric transformations can be represented by square transformation matrices only through homogeneous coordinates, an improved representation for geometric transformations better be based on homogeneous coordinates in projective space.

For geometric transformations which can be represented by square matrices, we have the following conclusion:

Theorem 0. *The different transformation matrices in different reference coordinate systems of the same geometric transformation are similar matrices.*

Proof. Suppose a geometric transformation $\mathscr{T}_0$ in projective space, which transforms an arbitrary point or hyperplane X into Y; and the homogeneous coordinates of X and Y in reference

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coordinate systems (I) and (II) are (x), $(x)'$, (y), $(y)'$ respectively; and the coordinate transformation matrix from the reference coordinate system (I) to (II) is a nonsingular matrix $\boldsymbol{T}$, i.e., $(x)' = \boldsymbol{T}(x)$, $(y)' = \boldsymbol{T}(y)$; Suppose the transformation matrices of $\mathscr{T}_0$ in different reference coordinate systems (I) and (II) are $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively, i.e., $(y) = \boldsymbol{A}(x)$, $(y)' = \boldsymbol{B}(x)'$;. Then we have:

$$(y)' = \boldsymbol{T}(y) = \boldsymbol{T}\,\boldsymbol{A}(x) = \boldsymbol{T}\,\boldsymbol{A}\,\boldsymbol{T}^{-1}(x)' \tag{1}$$

Since X is arbitrary, then $\boldsymbol{B} = \boldsymbol{T}\,\boldsymbol{A}\,\boldsymbol{T}^{-1}$, $\boldsymbol{B}$ and $\boldsymbol{A}$ are similar to each other. □

Theorem 0 indicates that, in order to define a geometric transformation reference coordinate system independently, we can take advantage of the inherent characteristics of the geometric transformation matrix, i.e., properties of the transformation matrix regarding eigen value(s), eigenvector(s), and the geometric meanings thereof.

Unless otherwise specified, the current work will be discussed only in the real field: $\mathbb{R}$, and a geometric transformations will be a point (*not hyperplane*) transformation discussed in *one* reference coordinate system. Therefore, $\mathscr{T}$ can be used to represent both a transformation and the transformation matrix thereof. Symbols like (s), (x),$(x)_i$, (π) and so on, represent the homogeneous coordinate *column* vectors for the point S, X, X_i, and hyperplane π and so on, respectively.

2. Stereohomology

2.1. Extension of Desargues theorem & Desarguesian configuration

Theorem 1 (Extended Desargues theorem [1, 2]). *In n- dimensional projective space* ($2 \leqslant n \in \mathbb{Z}^+$), *if the homogeneous coordinate vectors of the n+1 points* $X_1, X_2, ..., X_n, X_{n+1}$, *are linearly independent, any n homogeneous coordinate vectors of the n+1 points* $Y_1, Y_2, ..., Y_n, Y_{n+1}$, *are also linearly independent, and there exists a fixed point* S, *the homogeneous coordinate vector of which is linearly dependent to those of any two of the corresponding point pairs:* X_i *and* Y_i, ($i = 1, \cdots, n+1$). *Then the rank of the homogeneous coordinate vectors of the* C_{n+1}^2 *intersection points of line pairs* X_iX_j *and* Y_iY_j, *which will here be defined as* $S_{ij} = S_{ji} \overset{def}{=\!=} X_iX_j \cap Y_iY_j$ ($i \neq j$, $i, j = 1, \cdots, n+1$), *should be n.*

Proof. See [1]. □

Similar to the Desargues theorem and its inverse, the inverse of Theorem 1 also is true.

Definition 1 (Extended Desarguesian configuration). If $2n+3$ points, $X_1, X_2, \cdots, X_n, X_{n+1}, S, Y_1, Y_2, \cdots, Y_n, Y_{n+1}$, meet the constraints in both Theorem 1 and its inverse, then the configuration that consists of all these points is an Extended Desarguesian configuration, denoted as

$$X_1\,X_2 \cdots X_n\,X_{n+1}\text{-}S\text{-}Y_1Y_2 \cdots Y_n\,Y_{n+1} \tag{2}$$

2.2. Definition of stereohomology

Definition 2 (Generalized projective transformation). If the rank of the $n+1$ dimensional coefficient matrix $(t_{i,j})$ of the following transformation in n dimensional projective space is equal to or greater than n, then the transformation in projective space is called a generalized projective transformation ($0 \neq \rho \in \mathbb{R}$):

$$\begin{cases} \rho x_1' = t_{1,1}x_1 + t_{1,2}x_2 + \cdots + t_{1,n+1}x_{n+1} \\ \rho x_2' = t_{2,1}x_1 + t_{2,2}x_2 + \cdots + t_{2,n+1}x_{n+1} \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad \ddots \qquad \vdots \\ \rho x_{n+1}' = t_{n+1,1}x_1 + t_{n+1,2}x_2 + \cdots + t_{n+1,n+1}x_{n+1} \end{cases} \tag{3}$$

Definition 3 (invariant point). In projective space, if the homogeneous coordinate vector of a point is an eigenvector of a generalized projective transformation $\mathscr{T}$, then the point is called an invariant point of $\mathscr{T}$.

Definition 4 (General Invariant hyperplane). Suppose X is any point in hyperplane π, if its corresponding point X' through a generalized projective transformation matrix $\mathscr{T}$, is also in π, then π is a general invariant hyperplane of $\mathscr{T}$.

Definition 5 (Invariant hyperplane). Suppose X is any point in the general invariant hyperplane π of $\mathscr{T}$, if X's corresponding point X' coincides with X, then π is an invariant hyperplane of $\mathscr{T}$.

Definition 6 (null vector). If a generalized projective transformation $\mathscr{T}$ is singular, then there exists a point, denoted as X, of which the homogeneous coordinate vector, (x), will be transformed into $(0, \cdots, 0)^\top$, to which there is no corresponding geometric point in projective space, such a column homogeneous coordinate (x) is called a *null* vector for $\mathscr{T}$.

Lemma 1 (existence of "stereohomology center"). In n- dimensional projective space, if a generalized projective transformation $\mathscr{T}$ has an invariant hyperplane π, then there exists a unique fixed point, which is collinear with any point and its image point through $\mathscr{T}$; and when $\mathscr{T}$ is nonsingular, the fixed point is an invariant point of $\mathscr{T}$; when $\mathscr{T}$ is singular, the fixed point is the *null* vector of $\mathscr{T}$.

Proof. Denote the generalized projective transformation matrix as $\mathscr{T}$, and its invariant hyperplane as π. Suppose the homogeneous coordinate vector of any point X (denote its corresponding point through $\mathscr{T}$ as Y) can be linearly expressed by that of an arbitrary invariant point $H \in \pi(\rho_H \cdot (h) = \mathscr{T}(h)$, $0 \neq \rho_H \in \mathbb{R}$ is the eigenvalue for all the points in π) and that of a fixed point $C \notin \pi$ as:

$$(x) = (h) + \omega \cdot (c) \qquad \omega \in \mathbb{R} \tag{4}$$

therefore, Y's homogeneous coordinate can be:

$$(y) = \mathscr{T} \cdot (x) = \rho_H \cdot (h) + \omega \cdot \mathscr{T} \cdot (c)$$
$$\rho_H \in \mathbb{R}, \quad \text{and} \quad \omega \in \mathbb{R} \tag{5}$$

From "Eq.(5) – Eq.(4) × ρ_H ", we have:

$$(y) - \rho_H \cdot (x) = \omega \cdot (\mathscr{T} \cdot (c) - \rho_H \cdot (c))$$

which indicates that there exist a unique fixed point, denoted as S below(the *uniqueness* can be proved through *proof by contradiction*):

$$(s) = \mathscr{T} \cdot (c) - \rho_{H} \cdot (c),$$

which is collinear to X and the corresponding Y.

Since any line through S is transformed into itself, then S is an invariant point when $\mathscr{T}$ is nonsingular or S is the *null* vector when $\mathscr{T}$ is singular ($S \notin \pi$ in this case). □

Definition 7 (Stereohomology)**.** In n- dimensional projective space, a generalized projective transformation is called a stereohomology, if (1) the coefficient matrix thereof has a *stereohomology hyperplane* (denoted as π), the homogeneous coordinate vector of any point on which is an eigenvector of the transformation matrix, and the rank of the vector set with all these eigenvectors is n; and (2) there exists a fixed point (called *stereohomology center*, denoted as S), of which the homogeneous coordinate vector is linearly dependent to those of any point in the n- dimensional projective space, and its corresponding point through the generalized projective transformation.

According to *Lemma* 1, Definition 7 can actually be simplified as: a generalized projective transformation with an invariant hyperplane.

Definition 8. (Elementary homology) An elementary homology geometric transformation is a stereohomology, of which the stereohomology center is not on the stereohomology hyperplane, i.e., the inner product of the homogeneous coordinate vectors of the stereohomology center and the stereohomology hyperplane, is not zero.

Definition 9. (Elementary perspective) An elementary perspective is a stereohomology, of which the stereohomology center is on the stereohomology hyperplane.

Lemma 2 (Existence & uniqueness theorem). For the Extended Desarguesian Configuration (Eq.(2) in Definition 1), there exists a unique generalized projective transformation, denoted as $\mathscr{T}$, which transforms $X_1, X_2, \cdots X_n, X_{n+1}$ and S into $Y_1, Y_2, \cdots Y_n, Y_{n+1}$ and S (or *null*) respectively.

Proof. See [1]. □

The final form of the transformation matrix $\mathscr{T}^{3d}$ in 3- dimensional projective space, which transforms A, B, C, D, into A', B', C', D', and S into S (or *null*) in the extended Desarguesian configuration (Eq. (2) in Definition 1), respectively, will be obtained in the following form [1] (The Δ_is and Δ_i's are determinants defined in [1]):

$$\mathscr{T}^{3d} = k \times \begin{bmatrix} a_1'\Delta_1' & b_1'\Delta_2' & c_1'\Delta_3' & d_1'\Delta_4' \\ a_2'\Delta_1' & b_2'\Delta_2' & c_2'\Delta_3' & d_2'\Delta_4' \\ a_3'\Delta_1' & b_3'\Delta_2' & c_3'\Delta_3' & d_3'\Delta_4' \\ a_4'\Delta_1' & b_4'\Delta_2' & c_4'\Delta_3' & d_4'\Delta_4' \end{bmatrix}$$

$$\times \begin{bmatrix} a_1\Delta_1 & b_1\Delta_2 & c_1\Delta_3 & d_1\Delta_4 \\ a_2\Delta_1 & b_2\Delta_2 & c_2\Delta_3 & d_2\Delta_4 \\ a_3\Delta_1 & b_3\Delta_2 & c_3\Delta_3 & d_3\Delta_4 \\ a_4\Delta_1 & b_4\Delta_2 & c_4\Delta_3 & d_4\Delta_4 \end{bmatrix}^{-1} \tag{6}$$

The square transformation matrix of stereohomology should be obtained via more simple and direct method rather than that in (Eq.(6)) from the coordinate information of an Extended Desarguesian Configuration.

Lemma 3 (Existence of an n- dimensional Eigenspace). The obtained n+1 dimensional transformation matrix of generalized projective transformation $\mathscr{T}$ in Lemma 2, has an eigenvalue with geometric multiplicity of n, i.e., there exists an n- dimensional Eigenspace of the transformation matrix, and a hyperplance of the transformation $\mathscr{T}$.

Proof. Here only gives the proof when the transformation matrix is nonsingular, i.e., not only all the homogeneous coordinate vectors $(x)_i$ are linearly independent, but all the vectors $(y)_i$ are linearly independent. And the symbol $\mathscr{T}$ here will be used to represent both the *geometric transformation* and the *transformation matrix*.

To prove the current statement, we need to construct and use the properties of the minimal polynomial of the n+1-dimensinal transformation matrix $\mathscr{T}$.

First, since the homogeneous coordinate vector of S can be linearly expressed as:

$$\begin{aligned} (s) &= \lambda_1(x)_1 + \lambda_1'(y)_1 \\ &= \lambda_2(x)_2 + \lambda_2'(y)_2 \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \\ &= \lambda_{n+1}(x)_{n+1} + \lambda_{n+1}'(y)_{n+1} \end{aligned} \tag{7}$$

and since all the n+1 homogeneous coordinate vectors of either $(x)_i$ or $(y)_i$ ($i = 1, \cdots, n+1$) are linearly independent, $\exists\ 0 \neq \mu_i \in \mathbb{R}$ ($i = 1, \cdots, n+1$), which make (s) can be linearly expressed as:

$$(s) = \sum_{i=1}^{n+1} \mu_i\ (x)_i \tag{8}$$

Since $(x)_i$ and $(y)_i$ are the corresponding points through the geometric transformation $\mathscr{T}$ ($\forall\ i = 1, \cdots, n+1$), i.e., there exist a series of $0 \neq \rho_i \in \mathbb{R}$ ($i = 1, \ldots, n+1$), which satisfy:

$$\rho_i\,(y)_i = \mathscr{T}(x)_i \qquad (i = 1, \cdots, n+1) \tag{9}$$

$$\text{and} \quad \rho_s\,(s) = \mathscr{T}(s) \tag{10}$$

combine all the above results in Eq.(7, 8, 9, 10) together, we will obtain the linear expression of (s) by $(y)_i$ in two different forms:

$$\left(\sum_{i=1}^{n+1}\frac{\mu_i}{\lambda_i}-1\right)(s)=\sum_{i=1}^{n+1}\frac{\mu_i\lambda_i'}{\lambda_i}(y_i) \tag{11}$$

and

$$(s)=\frac{1}{\rho_s}\mathscr{T}\sum_{i=1}^{n+1}\mu_i(x)_i=\sum_{i=1}^{n+1}\frac{\rho_i\mu_i}{\rho_s}(y_i) \tag{12}$$

Since the linear expression of (s) by $(y)_i$ should be unique once the homogeneous coordinate vectors are selected, comparing the corresponding factors before each $(y)_i$, we can obtain:

$$\frac{\rho_i\lambda_i}{\lambda_i'}=\text{ const.}\qquad(\forall\ i=1,\cdots,n+1) \tag{13}$$

Since according to Eq. (10)

$$(s)=\frac{1}{\rho_s}\mathscr{T}\cdot(s) \tag{14}$$

and according to Eq.(7)

$$\begin{aligned}(s)&=\lambda_i(x)_i+\lambda_i'(y_i)\\&=\lambda_i(x)_i+\frac{\lambda_i'}{\rho_i}\mathscr{T}\cdot(x)_i\\&=\frac{\lambda_i}{\rho_s}\mathscr{T}\cdot(x)_i+\frac{\lambda_i'}{\rho_s\rho_i}\mathscr{T}^2\cdot(x)_i\\&\forall\ i=1,\cdots,n+1\end{aligned} \tag{15}$$

Then from Eq.(15) we can obtain:

$$(\mathscr{T}-\rho_s\boldsymbol{I})\left(\mathscr{T}+\frac{\rho_i\lambda_i}{\lambda_i'}\boldsymbol{I}\right)(x)_i=\mathbf{0}\qquad\forall\ i=1,\cdots,n+1 \tag{16}$$

Since ρ_s is an eigenvalue of the transformation matrix $\mathscr{T}$, and $\mathscr{T}\neq\boldsymbol{I}$,

$$(t-\rho_s)\left(t+\frac{\rho_i\lambda_i}{\lambda_i'}\right) \tag{17}$$

is the minimal characteristic polynomial of the transformation matrix $\mathscr{T}$. Consequently,

$$-\frac{\rho_i\lambda_i}{\lambda_i'}=\text{const.} \tag{18}$$

is also an eigenvalue of transformation matrix $\mathscr{T}$, and then it can be proved that the geometric multiplicity thereof is n.

Then prove that all the homogeneous coordinate vectors of the C_{n+1}^2 different intersection points, denoted as $(s)_{i,j}$, ($i\neq j$, i, j=1, $\cdots$, n+1), are the associated eigenvectors of the eigenvalue Eq. (18).

According to Eq. (7), which can be rewritten as:

$$(s)=\lambda_i(x)_i+\lambda_i'(y)_i\quad(i{=}1,\cdots,n{+}1)$$

and

$$\begin{cases}(s)_{1,2}=(x)_1-(x)_2=(y)_2-(y)_1\\(s)_{2,3}=(x)_2-(x)_3=(y)_3-(y)_2\\\quad\vdots\\(s)_{k,k+1}=(x)_k-(x)_{k+1}=(y)_{k+1}-(y)_k\\\quad\vdots\\(s)_{n,n+1}=(x)_n-(x)_{n+1}=(y)_{n+1}-(y_n)\\(s)_{n+1,1}=(x)_{n+1}-(x)_1=(y)_1-(y)_{n+1}\end{cases} \tag{19}$$

which can be rewritten as (i, j =1,$\cdots$, n+1):

$$\begin{aligned}(s)_{i,j}&=\lambda_i(x)_i-\lambda_j(x)_j\\&=\lambda_j'(y)_j-\lambda_i'(y)_i\end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathscr{T}\cdot(s)_{i,j}&=\mathscr{T}\cdot(\lambda_i(x)_i-\lambda_j(x)_j)\\&=\lambda_i\rho_i(y)_i-\lambda_j\rho_j(y)_j\end{aligned} \tag{21}$$

Comparing the linear expression of $(s)_{i,j}$ in Eq. (20) and $\mathscr{T}\cdot(s)_{i,j}$ in Eq. (21) by $(y)_j$ and $(y)_i$, considering Eq. (13), we thus can obtain that:

$$\mathscr{T}\cdot(s)_{i,j}=-\frac{\rho_i\lambda_i}{\lambda_i'}(s)_{i,j}\qquad\forall\ i\neq j,\quad i,j=1,\cdots,n+1 \tag{22}$$

Since the rank of the vector set which consists of all the homogeneous coordinate vectors of the C_{n+1}^2 intersection points is n, and all the vectors are associated eigenvectors, the geometric multiplicity of the associated eigenvalue is n.

When $\mathscr{T}$ is a singular matrix, the proof is similar with only slight difference. □

The above statement of *Lemma* 3 is actually corresponding to the extended Desargues theorem: Theorem 1; and *Lemma* 4 below will be corresponding to the inverse thereof.

Lemma 4 (Existence of “stereohomology center”). In n- dimensional real projective space, if a generalized projective transformation transforms the points $X_1, X_2,\cdots X_n$,and X_{n+1} in extended Desarguesian configuration (Eq.(2) in Definition 1)

into $Y_1, Y_2,\cdots Y_n$,and Y_{n+1} respectively, and there exists an eigenvalue with geometric multiplicity of n for the n+1 dimensional transformation matrix $\mathscr{T}$. Then the homogeneous coordinate vector of S is an eigenvector of $\mathscr{T}$, and the corresponding eigenvalue equals zero in case that $\mathscr{T}$ is singular.

Proof. Proof is omitted here. □

Lemma 5 (“Stereohomology center ”Linear correlationship). For the generalized projective transformation $\mathscr{T}$, which transforms $X_1, X_2,\cdots, X_n$, X_{n+1} and S in extended Desarguesian configuration (Eq.(2) in Definition 1)

into $Y_1, Y_2,\cdots, Y_n$, Y_{n+1} and S (or *Null*) respectively, for any point X, and its corresponding points Y through the transformation $\mathscr{T}$, the homogeneous coordinate vectors (x) and (y) of the two points are linearly dependent to (s).

Proof. Since there exists an invariant hyperplane π of the transformation per *Lemma* 3, according to *Lemma* 1, there exists a fixed invariant point (s) which is the stereohomology center of $\mathscr{T}$. Denote the eigenvalue corresponding to the invariant hyperplane π as λ, then the current statement is equivalent to: for any point X we have:

$$\exists\,\mu\in\mathbb{R}\ :\quad\mathscr{T}\cdot(x)=\lambda\cdot(x)+\mu\cdot(s) \tag{23}$$

Further proof will be omitted here. □

Theorem 2 (Existence & Uniqueness of stereohomology matrix)**.** *A stereohomology matrix can be uniquely determined by an extended Desarguesian configuration: the unique generalized projective transformation matrix which transforms* $X_1, X_2, \cdots X_n,\ X_{n+1}$ *and* S *in extended Desarguesian configuration* Eq. (2) *in Definition 1 into* $Y_1, Y_2, \cdots Y_n,\ Y_{n+1}$ *and* S *(or Null) respectively.*

Proof. First, a generalized projective transformation coefficient matrix determined by an Extended Desarguesian Configuration is a stereohomology, which can be proved from Lemmas.

The uniqueness of the generlized projective transformation therefore is equivalent to the uniqueness which has been proved in *Lemma* 2 □

Property 1. The order of the minimal characteristic polynomial of a stereohomology matrix always is 2.

Property 2. If an $(n+1)$- dimensional stereohomology matrix has two different eigenvalues, one of which should be the primary eigenvalue λ, and have geometric multiplicity of n, the other eigenvalue can be denoted by ρ; when $\lambda=\rho$, the stereohomology matrix can not be diagonalized.

Definition 10 (primary eigenvalue)**.** It has been proved that an $(n+1)$ dimensional stereohomology transformation matrix $\mathscr{T}$ has at least one eigenvalue, denoted as λ, with geometric multiplicity of n. This eigenvalue λ is called the primary eigenvalue of stereohomology $\mathscr{T}$.

3. Matix representation of stereohomology

Definition 11 (Elemetary Matrix)**.** An elementary matrix is a matrix which can be represented as [3]:

$$\begin{gathered}\boldsymbol{E}(\boldsymbol{u},\boldsymbol{v};\sigma)\overset{def}{=\!=}\boldsymbol{I}-\sigma\cdot\boldsymbol{u}\cdot\boldsymbol{v}^{\top}\\ \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^n,\quad 0\neq\sigma\in\mathbb{R}\end{gathered}\tag{24}$$

As has been mentioned that the stereohomology matrix obtained from Extended Desarguesian configuration, with the form in Eq. (6) is too complicated for application. And most commonly, we don't use the coordinate information from the Extended Desarguesian configuration for stereohomology related transformation determination. e.g., for those transformations which are intended to be included by stereohomology, it should be sufficient to determine a central projection, by coordinate information of a projection center and a projection plane, or to determine a parallel projection by projection direction and projection plane; and so forth.

These considerations finally lead to a rather simple representation for stereohomolgy: elementary matrix in numeric analysis, which begins with symbolically constructing the transformation matrix of a specific 3D stereohomology.

For a 3- dimensional *elementary homology* $\mathscr{T}^{3d}$ with stereohomology center S: $(s_1, s_2, s_3, s_4)^\top$, and stereohomology hyperplane π: $(a, b, c, d)^\top$, suppose neither two of a, b, c, and d are zero at the same time, and the two eigenvalues of $\mathscr{T}^{3d}$ are λ, corresponding to π with a geometric mutiplicity of 3, and ρ, corresponding to S.

Therefore, $(s_1, s_2, s_3, s_4)^\top$ is an associated eigenvector of ρ, and the linearly independent $(-b, a, 0, 0)^\top$, $(-c, 0, a, 0)^\top$, $(-d, 0, 0, a)^\top$ are the associated eigenvectors of λ, i.e.:

$$\begin{cases}\mathscr{T}^{3d}\cdot(s_1,s_2,s_3,s_4)^\top=\rho\cdot(s_1,s_2,s_3,s_4)^\top\\ \mathscr{T}^{3d}\cdot\ (-b,a,0,0)^\top\ =\lambda\cdot\ (-b,a,0,0)^\top\\ \mathscr{T}^{3d}\cdot\ (-c,0,a,0)^\top\ =\lambda\cdot\ (-c,0,a,0)^\top\\ \mathscr{T}^{3d}\cdot\ (-d,0,0,a)^\top\ =\lambda\cdot\ (-d,0,0,a)^\top\end{cases}\tag{25}$$

Eq.(25) can be rewritten as:

$$\mathscr{T}^{3d}=\begin{bmatrix}-\lambda b&-\lambda c&-\lambda d&\rho s_1\\ \lambda a&0&0&\rho s_2\\ 0&\lambda a&0&\rho s_3\\ 0&0&\lambda a&\rho s_4\end{bmatrix}\times\begin{bmatrix}-b&-c&-d&s_1\\ a&0&0&s_2\\ 0&a&0&s_3\\ 0&0&a&s_4\end{bmatrix}^{-1}=\frac{1}{as_1+bs_2+cs_3+ds_4}\times$$
$$\begin{bmatrix}\rho as_1+\lambda bs_2+\lambda cs_3+\lambda ds_4&bs_1(\rho-\lambda)&cs_1(\rho-\lambda)&ds_1(\rho-\lambda)\\ as_2(\rho-\lambda)&\lambda as_1+\rho bs_2+\lambda cs_3+\lambda ds_4&cs_2(\rho-\lambda)&ds_2(\rho-\lambda)\\ as_3(\rho-\lambda)&bs_3(\rho-\lambda)&\lambda as_1+\lambda bs_2+\rho cs_3+\lambda ds_4&ds_3(\rho-\lambda)\\ as_4(\rho-\lambda)&bs_4(\rho-\lambda)&cs_4(\rho-\lambda)&\lambda as_1+\lambda bs_2+\lambda cs_3+\rho ds_4\end{bmatrix}\tag{26}$$

Then Eq. (26) can be rewritten and extended to n- dimensional projective space as:

$$\begin{gathered}\mathscr{T}(\,(s),(\pi);\lambda,\rho)\ \overset{def}{=}\ \lambda\cdot\boldsymbol{I}+(\rho-\lambda)\cdot\frac{(s)\cdot(\pi)^\top}{(s)^\top\cdot(\pi)}\\ (s),(\pi)\in\mathbb{R}^{n+1},\ (s)^\top\cdot(\pi)\neq 0,\ \rho\in\mathbb{R}\end{gathered}\tag{27}$$

For *elementary perspective* transformation, similarly we can obtain (using any fixed point $(x)\notin\pi$ as auxiliary point/vector, and its corresponding point homogeneous vector per Eq. (23) in *Lemma* 5):

$$\mathscr{T}((s),(\pi);(x),\lambda,\mu') \overset{def}{=} \lambda\cdot \boldsymbol{I} + \mu'\cdot\frac{(s)\cdot(\pi)^{\top}}{(x)^{\top}\cdot(\pi)}$$
$$(s),(\pi),(x)\in\mathbb{R}^{n+1}, (s)^{\top}\cdot(\pi)=0,$$
$$(x)^{\top}\cdot(\pi)\neq 0, \mu'\neq 0, (x)\text{satisfies Eq. (23)} \tag{28}$$

Eq. (28) can also be rewritten as:

$$\mathscr{T}((s),(\pi);\lambda,\mu)\overset{def}{=}\lambda\cdot\boldsymbol{I}+\frac{\mu\cdot(s)\cdot\pi^{\top}}{\sqrt{(s)^{\top}\cdot(s)\cdot(\pi)^{\top}\cdot(\pi)}}$$
$$(s),(\pi)\in\mathbb{R}^{n+1},\ (s)^{\top}\cdot(\pi)=0,\ 0\neq\mu\in\mathbb{R} \tag{29}$$

In Eq. (29), the $\sqrt{(s)^{\top}\cdot(s)\cdot(\pi)^{\top}\cdot(\pi)}$ in the denominator is added only in order that the parameter μ' can be independent to the homogeneous coordinates selection for (s) and (π).

The transformation matrices constructed have generality for any other elementary homology or elementary perspective respectively.

Lemma 6 (Sylvester theorem). If a matrix $\boldsymbol{A}\in\mathbb{F}^{m\times n}$, $\boldsymbol{B}\in\mathbb{F}^{n\times m}$, and the characteristic polynomials of $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are $f_{AB}(\lambda)$ and $f_{BA}(\lambda)$ respectively, then:

$$f_{AB}(\lambda)=\lambda^{m-n}\cdot f_{BA}(\lambda) \tag{30}$$

Property 3 (rank of stereohomology). The rank of (n+1)- dimensional stereohomology matrix should be greater than or equal to n.

Proof. Let $\boldsymbol{A}=\alpha\cdot(s)$, $\boldsymbol{B}=(\pi)^{\top}$ $(\forall\,\alpha\in\mathbb{R})$. Applying Eq. (30) in *Lemma* 6 to the characteristic polynomials of $\boldsymbol{AB}$ and $\boldsymbol{BA}$, we have the following results:

When $S\notin\pi$, using Eq. (27) to represent $\mathscr{T}$. let

$$\alpha=-\frac{\rho-\lambda}{(s)^{\top}\cdot(\pi)}$$

then we have the determinant of stereohomology transformation matrix:

$$\det(\mathscr{T})=\det\left(\lambda\cdot\boldsymbol{I}+(\rho-\lambda)\frac{(s)\cdot(\pi)^{\top}}{(s)^{\top}\cdot(\pi)}\right)$$
$$=f_{AB}(\lambda)=\lambda^{n}\cdot f_{BA}(\lambda)=\lambda^{n}\cdot\rho \tag{31}$$

Since $\lambda\neq 0$ is the eigenvalue with the geometric multiplicity of n, if and only if $\rho=0$, we have rank$(\mathscr{T})=n$; otherwise rank$(\mathscr{T})=n$+1.

When $S\in\pi$, let

$$\alpha=\frac{-\mu}{\sqrt{(s)^{\top}\cdot(s)\cdot(\pi)(\pi)}}$$

the determinant of $\mathscr{T}$:

$$\det(\mathscr{T})=\det\left(\lambda\cdot\boldsymbol{I}+\mu\cdot\frac{(s)^{\top}\cdot(\pi)}{\sqrt{(s)^{\top}\cdot(s)\cdot(\pi)(\pi)}}\right)$$
$$=f_{AB}(\lambda)=\lambda^{n}\cdot f_{BA}(\lambda)=\lambda^{n+1} \tag{32}$$

Since λ is the only nonzero eigenvalue of $\mathscr{T}$, so for an elementary perspective transformation, $\mathscr{T}$ always is nonsingular. Here rank$(\mathscr{T})=n$+1. □

Theorem 3 (elementary matrix representation)**.** *A stereohomology geometric transformation, which is defined in* Definition *7, can and only can be represented as elementary matrix, which is defined by* Eq. (24) *in* Definition *11; and an elementary matrix always have the geometric meaning of stereohomology.*

Theorem 4 (tri-stereohomology theorem)**.** *If $\mathscr{T}_1$ and $\mathscr{T}_2$ are two different stereohomology transformation with stereohomology centers of S_1, S_2 respectively, and stereohomology hyperplanes of π_1 and π_2 respectively. $\mathscr{T}_3=\mathscr{T}_1\cdot\mathscr{T}_2$. Then:*

(i) *If S_1 coincides with S_2, then $\mathscr{T}_3$ is also a stereohomology; Denote the stereohomology hyperplane of $\mathscr{T}_3$ as π_3, then π_1, π_2 and π_3 are collinear.*

(ii) *If π_1 coincides with π_2, then $\mathscr{T}_3$ is also a stereohomology; Denote the stereohomology center of $\mathscr{T}_3$ as S_3, then S_1, S_2 and S_3 are collinear.*

Proof. Using Theorem 3, Eq. (27) and (29), and Definition 7, the current statement is straightforward. □

Unless otherwise specified, the primary eigenvalue of stereohomology λ, will be set as a default nonzero value of 1.

4. Applications of elementary matrices in computer graphics

The elementary matrix representation of stereohomology actually can be used for a series of geometric transformations which have been commonly used in computer graphics.

4.1. Represent geometric transformations by elementary matrices

Since some of the geometric transformations like projections, are actually singular transformations, while a projective transformation in projective geometry is nonsingular, in order to represent such kind of transformations, the concept of stereohomology is defined to be able to include both singular and nonsingular transformations.

Property 4 (singular stereohomology). The transformation matrix $\mathscr{T}$ of a singular stereohomology can and only can be represented as:

$$\mathscr{T}((s),(\pi))\overset{def}{=}\boldsymbol{I}-\frac{(s)\cdot(\pi)^{\top}}{(s)^{\top}\cdot(\pi)} \tag{33}$$

where $(s),(\pi)\in\mathbb{R}$, $(s)^{\top}\cdot(\pi)\neq 0$.

Definition 12 (reflexive; involutory)**.** If a projective geometric transformation $\mathscr{T}$ satisfies:

$$\mathscr{T}^{2}=k\cdot\boldsymbol{I},\quad \exists\, 0\neq k\in\mathbb{R}$$

then $\mathscr{T}$ is involutory, or is called an involutory (projective) transformation.

Property 5 (involutory stereohomology). An involutory stereohomology $\mathscr{T}$ can and only can be represented as:

$$\mathscr{T}((s),(\pi))\overset{def}{=}\boldsymbol{I}-2\times\frac{(s)\cdot(\pi)^{\top}}{(s)^{\top}\cdot(\pi)} \tag{34}$$

Definition 13 (Central projection)**.** A central projection $\mathscr{T}$ is a singular stereohomology, of which both the stereohomology center S and the stereohomology hyperplane π are at finity.

And the stereohomology center S is called the *projection center* of Central projection $\mathscr{T}$, and π is called the *projection hyperplane* or *image hyperplane* of $\mathscr{T}$.

Definition 14. (normal direction for finite hyperplane) For a point at infinity P_∞ and a hyperplane at finity π, if the inner product of the homogeneous coordinate vector of P_∞ and that of any infinite point in hyperplane π, is equal to zero, then we say P_∞ is the normal direction of π, or P_∞ is (projectively) orthogonal to π, denoted as $P_\infty \perp \pi$.

Specifically, in 3- dimensional projective space, if we use $(x_1, x_2, x_3, 0)^\top$ (where $x_1^2 + x_2^2 + x_3^2 \neq 0$) to represent the homogeneous coordinate vector of a point at infinity (***unless otherwise specified, all the examples discussed in 3-dimensional projective space in the present work will take this as a premise***), and if the homogeneous coordinate vector of a finite hyperplane π is $(a, b, c, d)^\top$, then the normal direction of π can be represented as $k \cdot (a, b, c, 0)^\top$ (where $0 \neq k \in \mathbb{R}$).

Definition 15 (Projectively parallel)**.** If there exists one intersection point at infinity of line l_1 and another line l_2, or hyperplane π, then we say line l_1 is projectively parallel, or simply, parallel, to line l_2, or hyperplane π, denoted as $l_1 \| l_2$, or $l_1 \| \pi$.

If there exists one intersection line at infinity of two hyperplanes π_1 and π_2, then we say hyperplane π_1 is projectively parallel, or simply, parallel to π_2, denoted as $\pi_1 \| \pi_2$.

The following conclusion will be straightforward:

Property 6 (normal direction property). If two finite hyperplanes $\pi_1 \parallel \pi_2$, then π_1 and π_2 have the same normal direction.

Definition 16 (Parallel projection)**.** A parallel projeciton $\mathscr{T}$ is a singular stereohomology, of which the stereohomology center S is at infinity, while the stereohomology hyperplane π is at finity.

The stereohomology center S is called the projection direction of parallel projection $\mathscr{T}$, and the stereohomology hyperplane π is called the projection hyperplane, or the image hyperplane of the parallel projection $\mathscr{T}$.

Specially, if $S \perp \pi$, the parallel projection $\mathscr{T}$ is called an orthogonal parallel projection.

An orthogonal parallel projection can be uniquely determined by its projection hyperplane.

Definition 17 (Reflection)**.** A general reflection $\mathscr{T}$ is an involutory stereohomology, of which, the stereohomology center S is at infinity, and the stereohomology hyperplane π is at finity. π is called the reflection hyperplane; and S is called the reflection direction.

Specially, when $S \perp \pi$, the general reflection $\mathscr{T}$ is called an orthogonal reflection, or simply called a *reflection*.

Definition 18 (Centrosymmetry)**.** A centrosymmetry transformation $\mathscr{T}$ is an involutory stereohomology, of which the stereohomology center S is at finity, while the stereohomolgy hyperplane π is at infinity.

S is called the symmetric center of $\mathscr{T}$.

Definition 19 (translation transformation)**.** A translation transformation is an elementary perspective, of which both the stereohomology center S and the stereohomology hyperplane π are at infinity.

But Definition 19 is not the best definition for a translation transformation especially when determining the transformation matrix. It better be defined as:

Definition 20 (translation transformaiton)**.** If two reflection $\mathscr{T}_1$ and $\mathscr{T}_2$, have parallel reflection hyperplanes $\pi_1 \parallel \pi_2$, then the compound transformation of two reflection $\mathscr{T}_1$ and $\mathscr{T}_2$: $\mathscr{T} = \mathscr{T}_1 \cdot \mathscr{T}_2$, is a translation transformation.

In Euclidean space, the translation distance of $\mathscr{T}$ is just twice that between π_1 and π_2. Or it can be defined as:

Definition 21 (translation transformation)**.** The compound transformation of two centrocymmetric transformation $\mathscr{T}_1$ and $\mathscr{T}_2$: $\mathscr{T} = \mathscr{T}_1 \cdot \mathscr{T}_2$, is a translation transformation.

In Euclidean space, the translation distance of $\mathscr{T}$ is twice that between the two symmetric centers S_1 and S_2; and translation direction can be determined according to the direction of oriented line $\overrightarrow{S_1 S_2}$.

Definition 22 (rotation transformation)**.** Suppose $\mathscr{T}_1$ and $\mathscr{T}_2$ are two *reflection* transformations, and the intersection line of the two reflection hyperplanes is: $l = \pi_1 \cap \pi_s$. The the compound transformation $\mathscr{T} = \mathscr{T}_1 \cdot \mathscr{T}_2$, is called a rotation transformation. l is the rotational axis of $\mathscr{T}$.

In Euclidean space, the rotation angle degree is twice that of the dihedral angle between π_1 and π_2.

4.2. Represent 3D reconstructing objects from multiple projections

4.2.1. A brief introduction

In *Vision*[4], Marr presented a very general discussion of representation and process for 3D reconstruction, and described three levels of information processing: theory, representation / algorithm, implementation.

In the present work, though the 3- dimensional reconstruction may mean the same thing: to reconstruct objectives from their images or projections, the basic concepts are still different from the conventional ones.

First, let us consider an axiom, or a speculation in the current representation, which is useful in 3D reconstruction but may be easily neglected by the conventional representation.

Speculation 1 (Feasibility of 3D reconstruction)**.** *A 3- dimensional reconstruction is feasible, if and only if, there exists a one-to-one mapping between objectives and their image consequence(s).*

Speculation 1 is true no matter what kind of 3D reconstruction one considers, i.e., either from multiple projections or from only single view, either by geometric optical principles or any other, with any additional constraints like symmetry and so forth or not.

From this point of view, the constraints in 3D reconstruction better be divided into two types, though it is still a little hard to present rigorous definitions for them:

Type I *Geometric constraints*, which are general for all kind of 3D reconstruction problems; e.g., the epipolar geometry constraints in 3D reconstruction; the Desargues theorem; and so forth;

Type II *Transcendental constraints*, which are special and only feasible for some specific cases; e.g., the symmetry of specific objectives, light and shadow, or color, and so on, of objectives.

4.2.2. Some basic definitions for the current representation

Definition 23 (normalized homogeneous coordinate)**.** Such homogeneous coordinate vector as $(x_1,\ x_2,\ x_3,\ 1)^\top$, which represents a point, is called normalized homogeneous coordinate, since which can be simply and directly mapping to $(x_1, x_2,\ x_3)^\top$ in Euclidean space.

For the homogeneous coordinate vector of an arbitrary point X': $(x'_1,\ x'_2,\ x'_3,\ x'_4)^\top$ (where $x'_4 \neq 0$, so that X' can be mapped into Euclidean space), the normalized homogeneous coordinate can be obtained by dividing x'_4 for every x'_i ($i = 1, \cdots, 4$).

The normalized homogeneous coordinate vector of (x) is denoted as $(x)^{\ominus}$

Definition 24 (normalization operation)**.** A normalization operation is a mapping which can map homogeneous coordinate vector, or homogeneous coordinate vector block matrix, into its normalized form. A normalized operation is denoted as: $(x)^{\ominus} = \mathscr{S}[(x)]\ X^{\ominus} = \mathscr{S}[X]$

Definition 25 (matrix dot multiplication)**.** If $m \times n$ dimensional matrices $\boldsymbol{A} = (a_{\mathrm{i},j})_{m\times n}$, $\boldsymbol{B} = (b_{\mathrm{i},j})_{m\times n}$, then the dot multiplication product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as:

$$\boldsymbol{A} \odot \boldsymbol{B} \overset{def}{=} \left(a_{\mathrm{i},j} \cdot b_{\mathrm{i},j}\right)_{m\times n} \tag{35}$$

Definition 26 (camera matrix)**.** In the current work, a central projection or parallel projection will be considered as a camera; and the transformation matrix thereof is called a camera matrix, denoted by $\mathscr{C}$.

According to definition, a camera matrix can be represented by Eq. (33).

According to Definition 26, a camera matrix represented by Eq. (33) for central projection camera matrices and parallel projection, is only a linear camera model. In the present work, the nonlinear distortion will not be considered for cameras.

Lemma 7 (*Null* Vector for Singular Stereohomology). For singular stereohomology defined by Eq. (33), and any non-null homogeneous coordinate column vector (x), if and only if $\exists\ 0 \neq k \in \mathbb{R}$, $(x) = k \cdot (s)$: $\mathscr{T} \cdot (x) = \mathbf{0}$

Definition 27 (Camera group matrix)**.** Suppose in 3- dimensional projective space, there are m ($m \geqslant 2$) different views of an objective projected by the following m different camera matrices in Eq. (36). Then Eq. (37) defines the camera group matrix, which is actually a block matrix (also can be called *multi-projection matrix*) which can be partitioned into the m different camera matrices(also called *sub-multi-projection matrices*).

$$\left.\begin{aligned} \mathscr{C}_1\left((s)_1,(\pi)_1\right) &\overset{def}{=} \boldsymbol{I} - \frac{(s)_1(\pi)_1^\top}{(s)_1^\top(\pi)_1} \\ \mathscr{C}_2\left((s)_2,(\pi)_2\right) &\overset{def}{=} \boldsymbol{I} - \frac{(s)_2(\pi)_2^\top}{(s)_2^\top(\pi)_2} \\ &\vdots \\ \mathscr{C}_m\left((s)_m,(\pi)_m\right) &\overset{def}{=} \boldsymbol{I} - \frac{(s)_m(\pi)_m^\top}{(s)_m^\top(\pi)_m} \end{aligned}\right\} \tag{36}$$

$$\mathscr{P} \overset{def}{=} \begin{bmatrix} \left[\boldsymbol{I} - \frac{(s)_1 \cdot (\pi)_1^\top}{(s)_1^\top \cdot (\pi)_1}\right] \\ \left[\boldsymbol{I} - \frac{(s)_2 \cdot (\pi)_2^\top}{(s)_2^\top \cdot (\pi)_2}\right] \\ \vdots \\ \left[\boldsymbol{I} - \frac{(s)_m \cdot (\pi)_m^\top}{(s)_m^\top \cdot (\pi)_m}\right] \end{bmatrix}_{4m\times 4} \tag{37}$$

Definition 28 (Camera Calibration)**.** In this work, the calibration of a camera means, the process of finding the homogeneous coordinate vectors of stereohomology center and the stereohomology hyperplane of a camera matrix.

Similarly, the calibration of a camera group matrix(e.g., defined by Eq. (37) in Definition 27), should be the process of finding all the homogeneous coordinate information of all the m pair of stereohomology centers and sterehomology hyperplanes for the m different camera matrices.

Definition 29 (objective matrix)**.** An objective matrix is a matrix, each column of which is a homogeneous coordinate column vector of a point, denoted by Ξ. Generally, an objective matrix representing n different points is a $4 \times n$ dimensional matrix. If all the homogeneous coordinate vectors in an objective matrix are normalized homogeneous coordinates, then the objective matrix is called a normalized objective matrix, denoted by $\Xi^{\ominus}$.

According to definition, $\Xi^{\ominus} = \mathscr{S}[\ \Xi\ ]$. and there exists a diagonal matrix Λ, which is called a *relaxation* matrix, and has the dimension of $n \times n$ to the image matrix Ξ, and which makes:

$$\Xi = \Xi^{\ominus} \cdot \Lambda = \mathscr{S}[\ \Xi\ ] \cdot \Lambda \tag{38}$$

Definition 30 (Subimage matrix)**.** An objective matrix Ξ or $\Xi^{\ominus}$ transformed by a sub-multi-projection matrix or camera matrix $\mathscr{C}$ will lead to a subimage matrix, denoted as ψ. If all the image point coordinates in ψ have been normalized, denoted as $\psi^{\ominus} = \mathscr{S}[\psi]$. Then:

$$\psi \overset{def}{=} \mathscr{C} \cdot \Xi \qquad \psi \overset{def}{=} \mathscr{C} \cdot \Xi \tag{39}$$

$$\psi^{\ominus} \cdot \Lambda \overset{def}{=} \mathscr{C} \cdot \Xi \qquad \psi^{\ominus} \cdot \Lambda \overset{def}{=} \mathscr{C} \cdot \Xi^{\ominus} \tag{40}$$

The Λ matrix is an $n \times n$ diagonal matrix. A subimage matrix has the same dimension to the corresponding $4 \times n$ objective matrix.

Definition 31 (image matrix)**.** An image matrix Ψ is a block matrix which consists of m different subimage matrices ψ_i ($i = 1, \cdots, m$), which can be obtained from transforming the $4 \times n$ objective matrix Ξ or $\Xi^{\ominus}$ by the $4m \times 4$ camera group matrix/multi-projection matrix $\mathscr{P}$. If all the m subimage matrices ψ_i are normalized, then the image matrix is called a normalized image matrix, denoted as $\Psi^{\ominus}$.

An image matrix can be defined by:

$$\Psi \overset{def}{=} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{bmatrix} = \begin{bmatrix} \mathscr{C}_1 \cdot \Xi \\ \mathscr{C}_2 \cdot \Xi \\ \vdots \\ \mathscr{C}_m \cdot \Xi \end{bmatrix} = \begin{bmatrix} \mathscr{C}_1 \\ \mathscr{C}_2 \\ \vdots \\ \mathscr{C}_m \end{bmatrix} \cdot \Xi = \mathscr{P} \cdot \Xi \tag{41}$$

According to definition, $\Psi^{\ominus} = \mathscr{S}[\ \Psi\ ]$, and there exists a *projective depth* matrix Γ, which has the same dimension to the image matrix Ψ and satisfies:

$$\Psi = \Gamma \odot \Psi^{\ominus} = \Gamma \odot \mathscr{S}[\ \Psi\ ] \tag{42}$$

Definition 32 (projective depth and projective depth matrix)**.** Since the projection of a point X into Y through camera matrix $\mathscr{C}$ can be represented by:

$$\gamma \cdot (y)^{\ominus} = \mathscr{C} \cdot (x)^{\ominus} \tag{43}$$

and n different points $X_1, X_2, \cdots, X_n$ into $Y_1, Y2, \cdots, Y_n$ through $\mathscr{C}$ can be represented by:

$$\begin{aligned} & C \cdot \begin{bmatrix} (x)_1^{\ominus} & (x)_2^{\ominus} & \cdots & (x)_n^{\ominus} \end{bmatrix} \\ = & \begin{bmatrix} \gamma_1 \cdot (y)_1^{\ominus} & \gamma_2 \cdot (y)_2^{\ominus} & \cdots & \gamma_n \cdot (y)_n^{\ominus} \end{bmatrix} \\ = & \begin{bmatrix} (y)_1^{\ominus} & (y)_2^{\ominus} & \cdots & (y)_n^{\ominus} \end{bmatrix} \cdot \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_n \end{bmatrix} \\ = & \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix} \odot \begin{bmatrix} (y)_1^{\ominus} & (y)_2^{\ominus} & \cdots & (y)_n^{\ominus} \end{bmatrix} \end{aligned} \tag{44}$$

For each camera $\mathscr{C}_i$, denote the matrix dot multiplication matrix in Eq. (44) as Γ_i, and that for the multiple camera projection, as Γ, then we have the following equation:

$$\Gamma \odot \Psi^{\ominus} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_m \end{bmatrix} \odot \begin{bmatrix} \psi_1^{\ominus} \\ \psi_2^{\ominus} \\ \vdots \\ \psi_m^{\ominus} \end{bmatrix} = \begin{bmatrix} \mathscr{C}_1 \\ \mathscr{C}_2 \\ \vdots \\ \mathscr{C}_m \end{bmatrix} \cdot \Xi^{\ominus} \tag{45}$$

In the aforementioned equations Eq. (42), (43), (44) and (45), γ_i is called *projective depth*, and the corresponding matrices Γ and Γ_i are the *projective depth matrices* for multi-projection matrix and for camera matrices respectively.

Theorem 5 (Projective depth theorem)**.** *For a camera matrix $\mathscr{C}$, which has the image hyperplane π, and the projection through which can be represented by Eq.* (43) *and* (44)*, if all the points $(x)_i^{\ominus}$ are on the same hyperplane α, which satisfies $\alpha \| \pi$, then all the projective depth γ_i in Eq.* (44) *should be a constant.*

Proof. Theorem 5 can be proved by analyzing the fourth entity of each obtained image homogeneous coordinate vector $(y)_i$ in Eq. (44) since $(y)_i = \gamma_i \cdot (y)_i^{\ominus}$, and γ_i is actually the fourth entity of $(y)_i$ according to Definitions 23 and 24. □

Theorem 5 can be applied in the simplification for projective depth matrix and its matrix dot multiplication operation in reconstruction.

Theorem 6 (Coordinate transformation of camera matrix)**.** *If there exists a coordinate transformation $\mathscr{L}$, which transforms camera matrix $\mathscr{C}_1$ into $\mathscr{C}_2$. Then $\mathscr{C}_2 = \mathscr{L} \cdot \mathscr{C}_1 \cdot \left((\mathscr{L})^{\top} \right)^{-1}$.*

4.2.3. Statements of projection and reconstruction problems

Definition 33 (Statement of projection)**.** Therefore the projections of any n different points by m different cameras defined in Eq. (36) can thus be represented by:

$$\Psi_{4m \times n} = \mathscr{P}_{4m \times 4} \cdot \Xi_{4 \times n}^{\ominus} \tag{46}$$

Here since $\Psi_{4m \times n}$ can also be represented by Eq. (42), then we have:

$$\Gamma_{4m \times n} \odot \Psi_{4m \times n}^{\ominus} = \mathscr{P}_{4m \times 4} \cdot \Xi_{4 \times n}^{\ominus} \tag{47}$$

where $\Xi_{4 \times n}^{\ominus}$ represents the normalized objective matrix, of which each column is corresponding to a normalized homogeneous coordinate vector of an objective point, and $\Psi_{4m \times n}$ represents the image matrix, which consists of the homogeneous coordinates of the M different projections.

Lemma 8 (Reconstruction Lemma). For the projection model in Eq. (46) and Eq. (47), the statement of *reconstruction is feasible for both* Eq. (46) *and* Eq. (47), is equivalent to that, $\mathscr{P}$ is a column full-rank matrix, i.e., rank($\mathscr{P}$)=4.

Proof. Actually, according to Speculation 1, Eq. (46) and Eq. (47) already define a mapping from objective to the corresponding image sequences. So the problem is equivalent to that, whether there exists an inverse mapping for Eq. (46) or Eq. (47).

According to matrix analysis theory, if and only if rank($\mathscr{P}$)= =4, $\mathscr{P}$ has left inverse matrices, denoted as $\mathscr{P}^{+}$. Eq. (46) or Eq. (47) premultiplying $\mathscr{P}^{+}$ will obtain the objective matrix from the image matrix, i.e., there exists an inverse mapping from image matrix into the objective matrix.

Here we don't need to consider the Γ matrix for Eq. (47). □

Definition 34 (Statement of reconstruction)**.** Simply, the reconstruction equation of n points can be obtained from their projection equation Eq. (47), denote the left inverse matrix of camera group matrix $\mathscr{P}$ as $\mathscr{P}^{\dagger}$, we have:

$$\Xi^{\ominus} = \mathscr{P}^{\dagger} \cdot \left(\Gamma \odot \Psi^{\ominus} \right) \tag{48}$$

Theorem 7 (Feasibility of reconstruction)**.** *For the projection and reconstruction model in the present work, from Speculation 1 we can obtain the following corollaries: The reconstruction is feasible if and only if:*

(i) *when all the camera matrices in a camera group matrix are central projections, and there exists at least two projection centers which do not coincide with each other;*

(ii) *when all the camera matrices in a camera group matrix are parallel projections, and there exists at least two projection directions, which do not coincide with each other;*

(iii) *when there are both central projection and parallel projection camera matrices in a camera group matrix.*

Proof. To prove this statement, we only need to consider the case when the camera group matrix $\mathscr{P}$ has two different camera matrices as sub-projection matrices. Suppose they are $\mathscr{C}_1$ and $\mathscr{C}_2$ with stereohomology centers of $(s)_1$, $(s)_2$ and image hyperplanes of $(\pi)_1$, and $(\pi)_2$ respectively. Postmultiply an arbitrary non-zero homogeneous coordinate vector (x) to $\mathscr{P}$, the current statement is equivalent to that, $\forall(x)$: $\mathscr{P}\cdot(x)\neq \mathbf{0}$. Since

$$\mathscr{P}\cdot(x) = \begin{bmatrix}\mathscr{C}_1\\ \mathscr{C}_2\end{bmatrix}\cdot(x) = \begin{bmatrix}\mathscr{C}_1\cdot(x)\\ \mathscr{C}_2\cdot(x)\end{bmatrix} \tag{49}$$

Use Lemma 7 to the partitioned block matrix $\mathscr{C}_1\cdot(x)$ and $\mathscr{C}_2\cdot(x)$ in Eq. (49), the statement will be straightforward. □

Theorem 7 is applicable for all of the multiple projection reconstruction problems using central/parallel projection camera models with only **Type I** constraints.

By definition, both Eq. (46) and Eq. (47) represents compatible equations, therefore when $\mathscr{P}$ is column full-rank, the objective matrix obtained by premultiplying $\mathscr{P}^+$ to these equations, is a least norm solution, and the $\mathscr{P}^+$ is a least norm pseudo inverse matrix for $\mathscr{P}$. Due to perturbations, they usually are contradictory equations; then linear least square approximation approach is used,and the left-inverse now denoted as $\mathscr{P}^\dagger$ in Eq. (48) is the Moore-Penrose pseudo-inverse of the $\mathscr{P}$.

Different $\mathscr{P}^+$s mean different approximation approaches to the solution.

Usually when we begin to reconstruct an objective from its projections, the Γ matrix here may not be exactly equal to that in Eq. (47). What we really need is the objective matrix Ξ, and the normalized $\Xi^{\ominus}$ can be obtained by applying normalization operation to Ξ. Then the reconstruction equation can be rewritten as:

$$\Xi = \mathscr{P}^\dagger\cdot\left(\Gamma'\odot\Psi^{\ominus}\right) \tag{50}$$

$$\text{or}:\quad \widehat{\Xi} = \widehat{\mathscr{P}^\dagger}\cdot\left(\widehat{\Gamma}\odot\widehat{\Psi^{\ominus}}\right) \tag{51}$$

$$\text{then}:\quad \widehat{\Xi^{\ominus}} = \mathscr{S}[\widehat{\Xi}]$$

In Eq. (51), symbols like $\widehat{\Xi}$, $\widehat{\Psi^{\ominus}}$, $\widehat{\mathscr{P}^\dagger}$, $\widehat{\Gamma}$, with a “wide hat” are used to represent the corresponding Ξ, $\Psi^{\ominus}$, $\mathscr{P}^\dagger$, Γ with perturbation respectively, which may be from measurement, floating point roundoff, any kind of distortions, and so on.

4.2.4. *Special camera models for reconstruction simplification*

In some application circumstances, there are some camera group models, in which we can have the reconstruction work simplified.

Definition 35 (Simplified Camera group model **Type I**)**.** The **type I** simplified camera group matrix $\mathscr{P}$, consists of camera matrices $\mathscr{C}_i$ $(i = 1,\cdots,m)$, which have the common stereohomololgy hyperplane π, and the lines l_j $(j = 1,\cdots,C_m^2)$ across any two of the stereohomology centers of which, should be parallel to the common stereohomology hyperplane π.

Property 7 (idealized central projection model reconstruction). In a reconstruction problem, if all the different views are obtained from the **Type I** simplified camera group model, then the calculation of projective depth matrix Γ can be omitted.

Proof. The statement can be proved by using Theorem 5. □

Definition 36 (Simplified Camera group model **Type II**)**.** The **Type II** simplified camera group matrix $\mathscr{P}$, consists of camera matrices $\mathscr{C}_i$ $(i = 1,\cdots,m)$, which are all parallel projection matrices.

Property 8. In a reconstruction problem, if all the different views are obtained from the **Type II** simplified camera group model, then the calculation of projective depth matrix Γ can be omitted.

Proof. The statement can be proved by using Theorem 5. □

Definition 37 (Simplified Camera group model **Type III**)**.** The **Type III** simplified camera group matrix $\mathscr{P}$, consists of camera matrices $\mathscr{C}_i$ $(i = 1,\cdots,m)$, which are all orthogonal parallel projection matrices.

Property 9 (orthogonal parallel projection model reconstruction). In the 3D reconstruction problems based on a Type II camera group matrix model, first, the calculation of the projective depth matrix Γ can be omitted; second, the camera calibration can be greatly simplified.

The parallel projection model can be directly applied to CAD circumstances, in which cameras are orthogonal parallel projection, the relative positions of cameras are simple, and the Γ matrix is not necessary for reconstruction.

Conclusion

(1) The concept of *stereohomology* has been proposed reference coordinate system independently, which includes a series of commonly used geometric transformations of which the transformation matrices are elementary matrices.

(2) The elementary matrix representation of stereohomology can be employed to represent the processes of projection and reconstruction in computer vision. A novel linear representation mathematical model has been presented.

(3) The current representation for 3D reconstruction actually provides a possible axiomatic approach to reconstruction problems with only **TYPE I** constraints.

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