4.16.9 Cubic Uniform B-Spline by Subdivision

The approach to cubic B-splines by subdivision is similar to that of Section 4.16.3. We show how Chaikin's methods (Section 4.15) can be applied to the construction of a cubic uniform B-spline for a set of $n+1$ control points $\mathbf{P}_i$. The points are divided into overlapping groups of four points each, and each group is used to calculate, by subdivision, a PC that becomes a segment in the entire curve. These cubic segments have C^2 continuity. Since subdivision is a recursive process, we denote the control points obtained after the kth subdivision by $\mathbf{P}_i^k$. Thus, it makes sense to denote the original control points by $\mathbf{P}_i^0$. They are divided into the overlapping groups

$$\mathbf{P}_0^0\mathbf{P}_1^0\mathbf{P}_2^0\mathbf{P}_3^0, \quad \mathbf{P}_1^0\mathbf{P}_2^0\mathbf{P}_3^0\mathbf{P}_4^0, \ldots, \mathbf{P}_{n-3}^0\mathbf{P}_{n-2}^0\mathbf{P}_{n-1}^0\mathbf{P}_n^0.$$

Figure 4.74a illustrates the refinement process that leads from the group of four control points $\mathbf{P}_0^0\mathbf{P}_1^0\mathbf{P}_2^0\mathbf{P}_3^0$ to a segment of a cubic uniform B-spline. The treatment for the other groups is similar. The figure shows the positions of the five iteration-1 points $\mathbf{P}_i^1$ and the seven points $\mathbf{P}_i^2$ resulting from iteration 2. The first refinement step computes the five points $\mathbf{P}_0^1\mathbf{P}_1^1\mathbf{P}_2^1\mathbf{P}_3^1\mathbf{P}_4^1$ as follows:

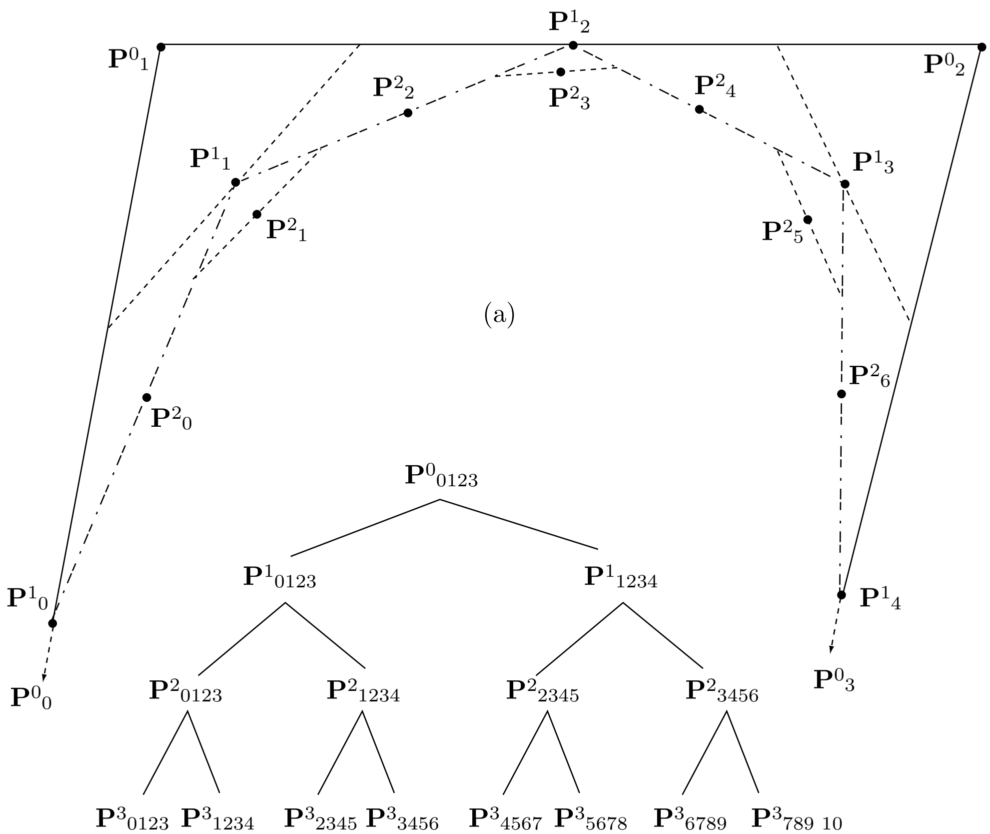
1. Each of the three points with even subscripts $\mathbf{P}_0^1\mathbf{P}_2^1\mathbf{P}_4^1$ (termed the *edge* points) is located at the center of a segment delimited by two of the original control points. Thus, $\mathbf{P}_0^1$ is located midway between $\mathbf{P}_0^0$ and $\mathbf{P}_1^0$.

2. Each of the two points with odd subscripts $\mathbf{P}_1^1$ and $\mathbf{P}_3^1$ (termed the *vertex* points) is located at the center of a segment whose endpoints are located at the centers of two segments delimited by two new edge points and one original control point. Thus, $\mathbf{P}_1^1$ is located at the center of the segment whose endpoints are located at the centers of the two segments delimited by the three points $\mathbf{P}_0^1$, $\mathbf{P}_1^0$ and $\mathbf{P}_2^1$.

The five points produced by the first refinement step can be expressed in terms of the four original control points by

$$\begin{pmatrix}\mathbf{P}_0^1\\ \mathbf{P}_1^1\\ \mathbf{P}_2^1\\ \mathbf{P}_3^1\\ \mathbf{P}_4^1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}(\mathbf{P}_0^0+\mathbf{P}_1^0)\\ \frac{1}{8}(\mathbf{P}_0^0+6\mathbf{P}_1^0+\mathbf{P}_2^0)\\ \frac{1}{2}(\mathbf{P}_1^0+\mathbf{P}_2^0)\\ \frac{1}{8}(\mathbf{P}_1^0+6\mathbf{P}_2^0+\mathbf{P}_3^0)\\ \frac{1}{2}(\mathbf{P}_2^0+\mathbf{P}_3^0)\end{pmatrix} = \frac{1}{8}\begin{pmatrix}4&4&0&0\\ 1&6&1&0\\ 0&4&4&0\\ 0&1&6&1\\ 0&0&4&4\end{pmatrix}\begin{pmatrix}\mathbf{P}_0^0\\ \mathbf{P}_1^0\\ \mathbf{P}_2^0\\ \mathbf{P}_3^0\end{pmatrix}.$$

Each of the new points $\mathbf{P}_i^1$ is computed from either two or three of the points $\mathbf{P}_j^0$. The five new points are then divided into two overlapping groups $\mathbf{P}_0^1\mathbf{P}_1^1\mathbf{P}_2^1\mathbf{P}_3^1$ and $\mathbf{P}_1^1\mathbf{P}_2^1\mathbf{P}_3^1\mathbf{P}_4^1$ of four points each, and the second subdivision step is applied to each group to produce five new points denoted by $\mathbf{P}_i^2$. Some of the $\mathbf{P}_i^2$ points, however, are identical, so this second step produces a total of seven points. Figure 4.74b shows the points produced by the first three iterations of the refinement process and how each group of four points $\mathbf{P}_i^k$ produces two overlapping groups of four new points $\mathbf{P}_i^{k+1}$ each. The compact notation $\mathbf{P}_{0123}^3$ stands for a group of four points. It is easy to see that iteration k produces 2^k overlapping groups of four points each, for a total of $4+(2^k-1)=3+2^k$ distinct



(b)

Figure 4.74: (a) The First Two Refinement Steps. (b) Groups After Three Steps.

points. Thus, iteration 0 (the original control points) consists of $3+2^0=4$ points, and iterations 1, 2, 3, and 4 produce 5, 7, 11, and 19 points, respectively.

Since each point produced in step k is computed from either two or three points of step $k-1$, it is convenient to express a new triplet of points $\mathbf{P}_i^k\mathbf{P}_{i+1}^k\mathbf{P}_{i+2}^k$ as a function of a triplet $\mathbf{P}_j^{k-1}\mathbf{P}_{j+1}^{k-1}\mathbf{P}_{j+2}^{k-1}$. We illustrate this relation for $k=1$

$$\begin{pmatrix}\mathbf{P}_0^1\\ \mathbf{P}_1^1\\ \mathbf{P}_2^1\end{pmatrix}=\mathbf{A}\begin{pmatrix}\mathbf{P}_0^0\\ \mathbf{P}_1^0\\ \mathbf{P}_2^0\end{pmatrix},\quad \begin{pmatrix}\mathbf{P}_2^1\\ \mathbf{P}_3^1\\ \mathbf{P}_4^1\end{pmatrix}=\mathbf{A}\begin{pmatrix}\mathbf{P}_1^0\\ \mathbf{P}_2^0\\ \mathbf{P}_3^0\end{pmatrix},\quad \text{where}\quad \mathbf{A}=\frac{1}{8}\begin{pmatrix}4&4&0\\ 1&6&1\\ 0&4&4\end{pmatrix},$$

or, using compact notation $\mathbf{P}_{012}^1=\mathbf{AP}_{012}^0$ and $\mathbf{P}_{234}^1=\mathbf{AP}_{123}^0$. In general $\mathbf{P}_{i\,i+1\,i+2}^1=\mathbf{AP}_{j\,j+1\,j+2}^0$ for even values of i and for $j=i,i-1$.

For $k=2$, the computation of the seven points $\mathbf{P}_i^2$ can be summarized by the three overlapping triplets $\mathbf{P}_{012}^2=\mathbf{AP}_{012}^1$, $\mathbf{P}_{234}^2=\mathbf{AP}_{123}^1$, and $\mathbf{P}_{456}^2=\mathbf{AP}_{234}^1$, or in general $\mathbf{P}^2_{i\,i+1\,i+2} = \mathbf{A}\mathbf{P}^1_{j\,j+1\,j+2}$, for even values of i and for $j = i$, $i-1$, and $i-2$. For $k = 3$, the calculation of the 11 points $\mathbf{P}^3_i$ is summarized by the five triplets $\mathbf{P}^3_{i\,i+1\,i+2} = \mathbf{A}\mathbf{P}^2_{j\,j+1\,j+2}$, where i is even and $j = i$, $i-1$, $i-2$, and $i-3$. In general, the computation of the $3+2^k$ points of step k can be summarized by the $2^{k-1}+1$ triplets $\mathbf{P}^k_{i\,i+1\,i+2} = \mathbf{A}\mathbf{P}^{k-1}_{j\,j+1\,j+2}$ where i is even and j goes through the values i, $i-1$ and so on, down to $i-(2^{k-1}-1)$.

◇ **Exercise 4.99:** Write each of the nine triplets $\mathbf{P}^4_{i\,i+1\,i+2}$ (for even values of i) in terms of a triplet $\mathbf{P}^3_{j\,j+1\,j+2}$.

Because of the repeated use of matrix $\mathbf{A}$, most triplets produced in step k can be expressed in terms of triplets produced in earlier steps. For example, the trio of points $\mathbf{P}^3_{012}$ can be written as $\mathbf{A}\mathbf{P}^2_{012} = \mathbf{A}^2\mathbf{P}^1_{012} = \mathbf{A}^3\mathbf{P}^0_{012}$, the triplet $\mathbf{P}^3_{234}$ equals $\mathbf{A}\mathbf{P}^2_{123}$, and $\mathbf{P}^3_{456}$ can be written as $\mathbf{A}\mathbf{P}^2_{234} = \mathbf{A}^2\mathbf{P}^1_{123}$. (Note that for the triplet on the left-hand side, the first subscript is always even, but the first subscript of the triplet on the right can be even or odd.) These relations point the way to moving forward from an earlier triplet to a later one. If we start, say, with the triplet $\mathbf{P}^1_{123}$, we can easily compute the triplets $\mathbf{P}^2_{234}$, $\mathbf{P}^3_{456}$, $\mathbf{P}^4_{89\,10}$, $\mathbf{P}^5_{16\,17\,18}$, and so on by multiplying the three points $\mathbf{P}^1_{123}$ by powers of $\mathbf{A}$. We can use this method to leapfrog across many recursion steps and proceed, in one step, from any triplet $\mathbf{P}^k_{i\,i+1\,i+2}$ to a triplet many subdivision steps later! In the limit, this can be written $\lim_{k\to\infty}\mathbf{P}^k_{i\,i+1\,i+2} = \mathbf{A}^\infty\mathbf{P}^k_{i\,i+1\,i+2}$, where $\mathbf{A}^\infty$ denotes $\lim_{k\to\infty}\mathbf{A}^k$. Any triplet $\mathbf{P}^k_{i\,i+1\,i+2}$ is an approximation to the ideal B-spline curve, but the limit $\lim_{k\to\infty}\mathbf{P}^k_{i\,i+1\,i+2}$ converges to a point on the actual curve.

The problem is therefore to calculate the limit of $\mathbf{A}^k$ as k approaches infinity, and this can easily be done with the help of the following theorem (see any text on matrices for the proof and for more information on eigenvalues and eigenvectors):

Theorem: If $\mathbf{A}$ is an $n\times n$ matrix for which there exist n linearly independent eigenvectors, then $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}$, where $\mathbf{Q}$ is the matrix whose columns are the n eigenvectors and Λ is the diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$.

This theorem implies that $\mathbf{A}^2 = \mathbf{Q}\Lambda\mathbf{Q}^{-1}\mathbf{Q}\Lambda\mathbf{Q}^{-1} = \mathbf{Q}\Lambda^2\mathbf{Q}^{-1}$, and in general $\mathbf{A}^k = \mathbf{Q}\Lambda^k\mathbf{Q}^{-1}$. Following this theorem, we can write our matrix $\mathbf{A}$ (after its eigenvalues and a set of linearly independent eigenvectors have been computed with appropriate software) as

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1\\ -1/2 & 0 & 1\\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/4 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & -2/3 & 1/3\\ -1/2 & 0 & 1/2\\ 1/6 & 2/3 & 1/6 \end{pmatrix}.$$

Since matrix Λ is diagonal, we have

$$\lim_{k\to\infty}\Lambda^k = \lim_{k\to\infty}\begin{pmatrix} (1/4)^k & 0 & 0\\ 0 & (1/2)^k & 0\\ 0 & 0 & 1^k \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

The limit $\mathbf{A}^\infty$ is therefore

$$\begin{pmatrix} 1 & -1 & 1 \\ -1/2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & -2/3 & 1/3 \\ -1/2 & 0 & 1/2 \\ 1/6 & 2/3 & 1/6 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 4 & 1 \\ 1 & 4 & 1 \\ 1 & 4 & 1 \end{pmatrix},$$

so we end up with the limits

$$\begin{aligned} \lim_{k\to\infty} \mathbf{P}^k_{i\,i+1\,i+2} &= \frac{1}{6}\begin{pmatrix} 1 & 4 & 1 \\ 1 & 4 & 1 \\ 1 & 4 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{P}^k_i \\ \mathbf{P}^k_{i+1} \\ \mathbf{P}^k_{i+2} \end{pmatrix} \stackrel{\text{def}}{=} \frac{1}{6}(1,4,1)\begin{pmatrix} \mathbf{P}^k_i \\ \mathbf{P}^k_{i+1} \\ \mathbf{P}^k_{i+2} \end{pmatrix} \\ &= \frac{1}{6}(\mathbf{P}^k_i + 4\mathbf{P}^k_{i+1} + \mathbf{P}^k_{i+2}), \end{aligned}$$

where k is any nonnegative integer. Notice that the three points of the triplet converge to the same point on the B-spline curve.

To summarize, we can (1) select four control points $\mathbf{P}^0_{0123}$, (2) select a value k and perform k refinement steps, (3) select a value i and a triplet $\mathbf{P}^k_{i\,i+1\,i+2}$, and (4) compute $(\mathbf{P}^k_i + 4\mathbf{P}^k_{i+1} + \mathbf{P}^k_{i+2})/6$. This will be a point on the cubic B-spline curve segment defined by the four original control points. To prove that this is so, we can express each of the three points $\mathbf{P}^k_{i\,i+1\,i+2}$ in terms of the original control points $\mathbf{P}^0_{0123}$, and compare the result with the general cubic B-spline segment, Equation (4.159). Here are some examples.

Example 1: We start with $k = 0$, $i = 0$. The initial triplet is therefore $\mathbf{P}^0_{012}$.

$$\lim_{k\to\infty} \mathbf{P}^0_{012} = \frac{1}{6}(1,4,1)\begin{pmatrix} \mathbf{P}^0_0 \\ \mathbf{P}^0_1 \\ \mathbf{P}^0_2 \end{pmatrix} = \frac{1}{6}(\mathbf{P}^0_0 + 4\mathbf{P}^0_1 + \mathbf{P}^0_2),$$

which is the initial point $\mathbf{P}(0)$ of the B-spline segment, as can be seen from Equation (4.159).

Example 2: The values $k = 0$, $i = 1$ specify the triplet $\mathbf{P}^0_{123}$ (notice that i does not have to be even).

$$\lim_{k\to\infty} \mathbf{P}^0_{123} = \frac{1}{6}(1,4,1)\begin{pmatrix} \mathbf{P}^0_1 \\ \mathbf{P}^0_2 \\ \mathbf{P}^0_3 \end{pmatrix} = \frac{1}{6}(\mathbf{P}^0_1 + 4\mathbf{P}^0_2 + \mathbf{P}^0_3),$$

which is the final point $\mathbf{P}(1)$ of the B-spline segment, as can be seen from the same equation.

Example 3: We perform one refinement steps and select the triplet $\mathbf{P}^1_{123}$ specified by $k = 1$ and $i = 1$. When this triplet is expressed in terms of the control points $\mathbf{P}^0_i$, the result is

$$\begin{aligned}
\lim_{k\to\infty} \mathbf{P}^1_{123} &= \frac{1}{6}(\mathbf{P}^1_1 + 4\mathbf{P}^1_2 + \mathbf{P}^1_3)\\
&= \frac{1}{6}\left(\frac{1}{8}(\mathbf{P}^0_0 + 6\mathbf{P}^0_1 + \mathbf{P}^0_2) + \frac{4}{2}(\mathbf{P}^0_1 + \mathbf{P}^0_2) + \frac{1}{8}(\mathbf{P}^0_1 + 6\mathbf{P}^0_2 + \mathbf{P}^0_3)\right)\\
&= \frac{1}{48}(\mathbf{P}^0_0 + 23\mathbf{P}^0_1 + 23\mathbf{P}^0_2 + \mathbf{P}^0_3).
\end{aligned}$$

Equation (4.159) tells us that this is the midpoint $\mathbf{P}(1/2)$ of the curve segment.

◇ **Exercise 4.100:** Select $k = 3$ and $i = 6$ and compute the point on the cubic B-spline curve segment obtained from these values at the limit of subdivision.

4.16.10 Higher-Degree Uniform B-Splines

The methods of Sections 4.16.2 and 4.16.4 can be used to compute uniform B-splines of higher degrees. It can be shown (see, e.g., [Yamaguchi 88], p. 329) that the degree-n uniform B-spline segment is given by

$$\mathbf{P}_i(t) = (t^n, \ldots, t^2, t, 1)\mathbf{M}\begin{pmatrix}\mathbf{P}_{i-1}\\ \mathbf{P}_i\\ \mathbf{P}_{i+1}\\ \vdots\\ \mathbf{P}_{i+n-1}\end{pmatrix},$$

where the elements m_{ij} of the basis matrix $\mathbf{M}$ are given by

$$m_{ij} = \frac{1}{n!}\binom{n}{i}\sum_{k=j}^{n}(n-k)^i(-1)^{k-j}\binom{n+1}{k-j}.$$

Figure 4.75 shows a few examples of these matrices.

4.16.11 Interpolating B-Splines

The B-spline is an approximating curve. Its shape is determined by the control points $\mathbf{P}_i$, but the curve itself does not pass through those points. Instead, it passes through the joints $\mathbf{K}_i$. In our notation so far, we have assumed that the cubic uniform B-spline is based on $n + 1$ control points and passes through $n - 1$ joint points. The number of control points for the cubic curve is, thus, always two more than the number of joints.

> One person's constant is another person's variable.
> — Susan Gerhart.

This section solves the opposite problem. We show how to use the B-spline method to calculate an interpolating cubic spline curve that passes through a set of $n+1$ given

$$
\begin{aligned}
&\quad + (-3t^3+3t^2+3t+1)(0,-3/2)+t^3(3/2,0)] \\
&= \frac{1}{4}(-2t^3+6t^2-4, 2t^3-6t), \\
\mathbf{P}_3(t) &= \frac{1}{6}[(-t^3+3t^2-3t+1)(-3/2,0)+(3t^3-6t^2+4)(0,-3/2) \\
&\quad + (-3t^3+3t^2+3t+1)(3/2,0)+t^3(0,3/2)] \\
&= \frac{1}{4}(-2t^3+6t, -2t^3+6t^2-4), \\
\mathbf{P}_4(t) &= \frac{1}{6}[(-t^3+3t^2-3t+1)(0,-3/2)+(3t^3-6t^2+4)(3/2,0) \\
&\quad + (-3t^3+3t^2+3t+1)(0,3/2)+t^3(-3/2,0)] \\
&= \frac{1}{4}(2t^3-6t^2+4, -2t^3+6t). \qquad \text{(Ans.33)}
\end{aligned}
$$

4.97: Compute the midpoint $(\mathbf{S}+\mathbf{E})/2$ and normalize its coordinates.

4.98: Equation (4.166) can be written $\mathbf{P}^t(t) = (t^2-t)[(\mathbf{P}_0-\mathbf{P}_3)+3(\mathbf{P}_1-\mathbf{P}_2)]$. This is the sum of two differences of points. The first difference is the vector from $\mathbf{P}_3$ to $\mathbf{P}_0$ and the second is the vector from $\mathbf{P}_2$ to $\mathbf{P}_1$ (multiplied by 3). The tangent vector of Equation (4.166) therefore points in the direction of the sum of these vectors, and this direction does not depend on t. The size of the tangent vector depends on t, but the size affects just the speed of the spline segment, not its shape.

4.99: This is straightforward and the triplets are $\mathbf{P}^4_{012} = \mathbf{A}\mathbf{P}^3_{012}$, $\mathbf{P}^4_{234} = \mathbf{A}\mathbf{P}^3_{123}$, $\mathbf{P}^4_{456} = \mathbf{A}\mathbf{P}^3_{234}$, $\mathbf{P}^4_{678} = \mathbf{A}\mathbf{P}^3_{345}$, $\mathbf{P}^4_{89\,10} = \mathbf{A}\mathbf{P}^3_{456}$, $\mathbf{P}^4_{10\,11\,12} = \mathbf{A}\mathbf{P}^3_{567}$, $\mathbf{P}^4_{12\,13\,14} = \mathbf{A}\mathbf{P}^3_{678}$, $\mathbf{P}^4_{14\,15\,16} = \mathbf{A}\mathbf{P}^3_{789}$, and $\mathbf{P}^4_{16\,17\,18} = \mathbf{A}\mathbf{P}^3_{89\,10}$.

4.100: The problem is to compute $\lim_{k\to\infty} \mathbf{P}^3_{678} = \frac{1}{6}(\mathbf{P}^3_6+4\mathbf{P}^3_7+\mathbf{P}^3_8)$. The *Mathematica* code of Figure Ans.27 does the calculations and produces the result $(\mathbf{P}^0_0 + 121\mathbf{P}^0_1 + 235\mathbf{P}^0_2 + 27\mathbf{P}^0_3)/384$. A comparison with Equation (4.159) shows that this is point $P(3/4)$ of the B-spline curve segment.

4.101: By substituting, for example, $t+1$ for t in the expression for $N_{13}(t)$.

4.102: The tangent vectors of the three segments are

$$
\begin{aligned}
\mathbf{P}^t_1(t) &= 2(t-1)\mathbf{P}_0 + (2-3t)\mathbf{P}_1 + t\mathbf{P}_2, \\
\mathbf{P}^t_2(t) &= (t-2)\mathbf{P}_1 + (3-2t)\mathbf{P}_2 + (t-1)\mathbf{P}_3, \\
\mathbf{P}^t_3(t) &= (t-3)\mathbf{P}_2 + (7-3t)\mathbf{P}_3 + 2(t-2)\mathbf{P}_4.
\end{aligned}
$$

They satisfy $\mathbf{P}^t_1(1) = \mathbf{P}^t_2(1) = \mathbf{P}_2 - \mathbf{P}_1$, and $\mathbf{P}^t_2(2) = \mathbf{P}^t_3(2) = \mathbf{P}_3 - \mathbf{P}_2$.

```
a={{4,4,0},{1,6,1},{0,4,4}}/8;
{p10,p11,p12}=a.{p00,p01,p02};
{p12,p13,p14}=a.{p01,p02,p03};
{p20,p21,p22}=a.{p10,p11,p12};
{p22,p23,p24}=a.{p11,p12,p13};
{p24,p25,p26}=a.{p12,p13,p14};
{p30,p31,p32}=a.{p20,p21,p22};
{p32,p33,p34}=a.{p21,p22,p23};
{p34,p35,p36}=a.{p22,p23,p24};
{p36,p37,p38}=a.{p23,p24,p25};
{p38,p39,p310}=a.{p24,p25,p26};
Simplify[(p36+4 p37+p38)/6]
```

Figure Ans.27: *Mathematica* Code For Exercise 4.100.

4.103: For the curve of Figure 4.85c, the knot vector is

$$(-3, -2, -1, 0, 1, 1, 1, 2, 3, 4, 5, 6).$$

The range of the parameter t is from $t_3 = 0$ to $t_8 = 3$ and we obtain the blending functions by direct calculations (only the last group N_{i4} of blending functions is shown):

$$\begin{aligned}
N_{04}(t) &= \frac{t-t_0}{t_3-t_0}N_{03} + \frac{t_4-t}{t_4-t_1}N_{13} = \frac{1}{6}(1-t)^3 && \text{for } t \in [0,1),\\
N_{14}(t) &= \frac{t-t_1}{t_4-t_1}N_{13} + \frac{t_5-t}{t_5-t_2}N_{23} = \frac{1}{12}(11t^3-15t^2-3t+7) && \text{for } t \in [0,1),\\
N_{24}(t) &= \frac{t-t_2}{t_5-t_2}N_{23} + \frac{t_6-t}{t_6-t_3}N_{33} = \frac{1}{4}(-7t^3+3t^2+3t+1) && \text{for } t \in [0,1),\\
N_{34}(t) &= \frac{t-t_3}{t_6-t_3}N_{33} + \frac{t_7-t}{t_7-t_4}N_{43} = \begin{cases} t^3 & \text{for } t \in [0,1),\\ (2-t)^3 & \text{for } t \in [1,2),\end{cases}\\
N_{44}(t) &= \frac{t-t_4}{t_7-t_4}N_{43} + \frac{t_8-t}{t_8-t_5}N_{53} = \frac{1}{4}\begin{cases} (7t^3-39t^2+69t-37) & \text{for } t \in [1,2),\\ (3-t)^3 & \text{for } t \in [2,3),\end{cases}\\
N_{54}(t) &= \frac{t-t_5}{t_8-t_5}N_{53} + \frac{t_9-t}{t_9-t_6}N_{63} = \frac{1}{12}\begin{cases} (-11t^3+51t^2-69t+29) & \text{for } t \in [1,2),\\ (7t^3-57t^2+147t-115) & \text{for } t \in [2,3),\end{cases}\\
N_{64}(t) &= \frac{t-t_6}{t_9-t_6}N_{63} + \frac{t_{10}-t}{t_{10}-t_7}N_{73} = \frac{1}{6}\begin{cases} (t-1)^3 & \text{for } t \in [1,2),\\ (-3t^3+21t^2-45t+31) & \text{for } t \in [2,3),\end{cases}\\
N_{74}(t) &= \frac{t-t_7}{t_{10}-t_7}N_{73} + \frac{t_{11}-t}{t_{11}-t_8}N_{83} = \frac{1}{6}(t-2)^3 && \text{for } t \in [2,3).
\end{aligned}$$

This group of blending functions can now be used to construct the five spline segments

$$\mathbf{P}_3(t) = N_{04}(t)\mathbf{P}_0 + N_{14}(t)\mathbf{P}_1 + N_{24}(t)\mathbf{P}_2 + N_{34}(t)\mathbf{P}_3$$